Gravitational Instability

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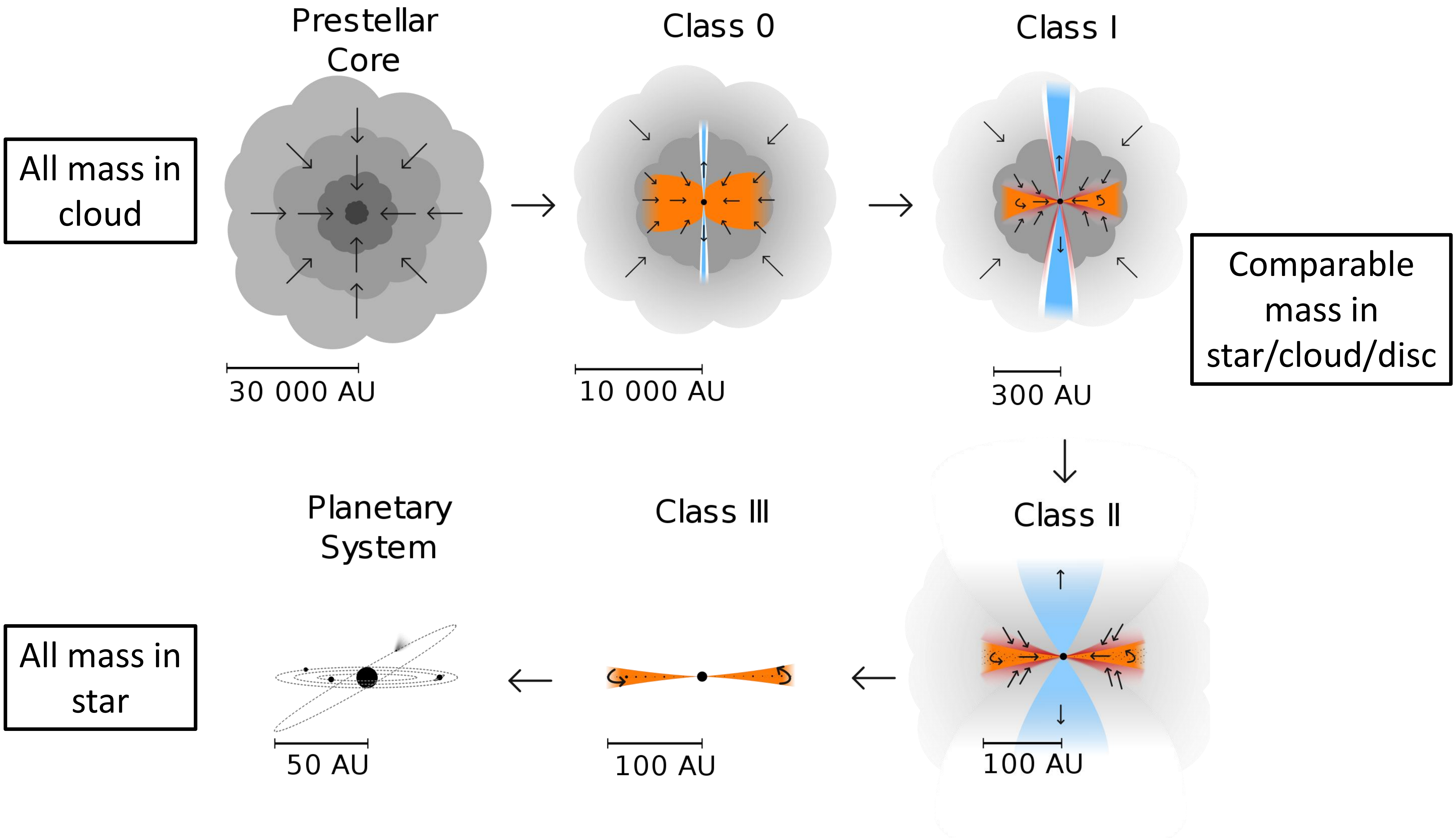


All mass in cloud
Prestellar Core
30 000 AU
Class 0
10 000 AU
Class I
300 AU
Comparable mass in star/cloud/disc
Class II
100 AU
Class III
100 AU
Planetary System
50 AU
All mass in star

When are discs gravitationally unstable?

- Toomre's Q parameter
- Ratio of stabilizing to destabilizing forces
  - $Q < 1$: Instability

Pressure

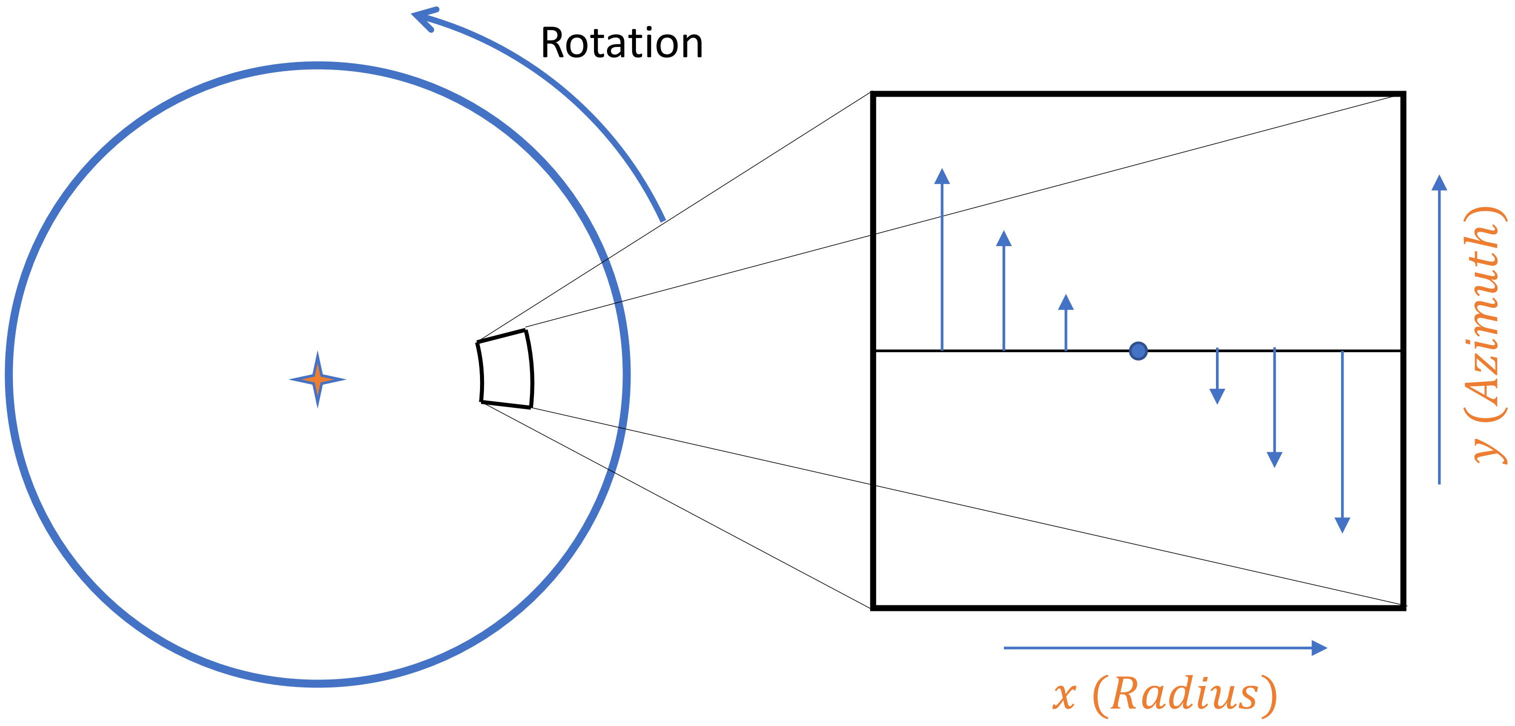
Rotation

$$Q = \frac{c_s \Omega}{\pi G \Sigma}$$

Gravity

Origin of the Toomre Q

Shearing-box Geometry

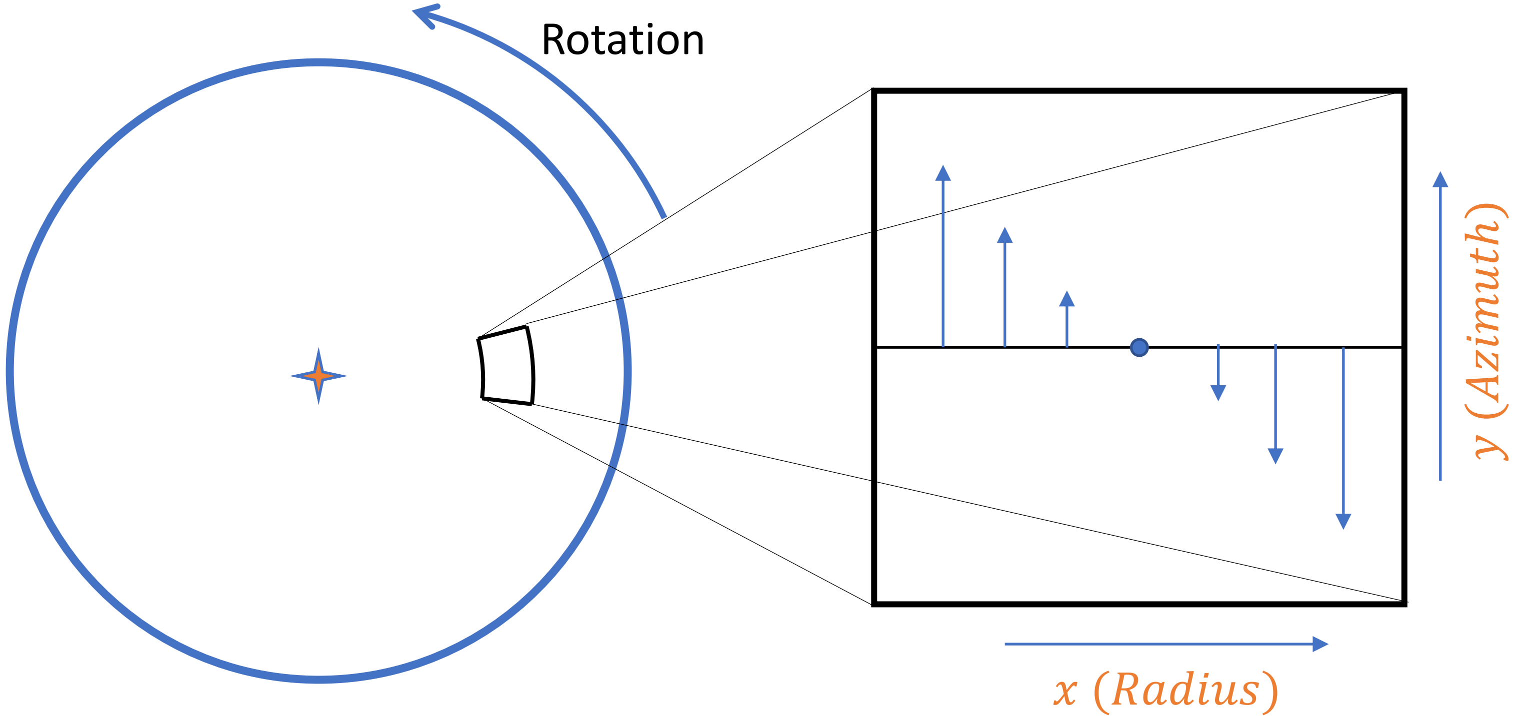


Shearing Box Model:

- Small patch of disc
- Rotates with the fluid
- $\frac{d\Omega}{dr}$ introduces *shear*
- Also: *thin disc approximation*

Origin of the Toomre Q

Shearing-box Geometry

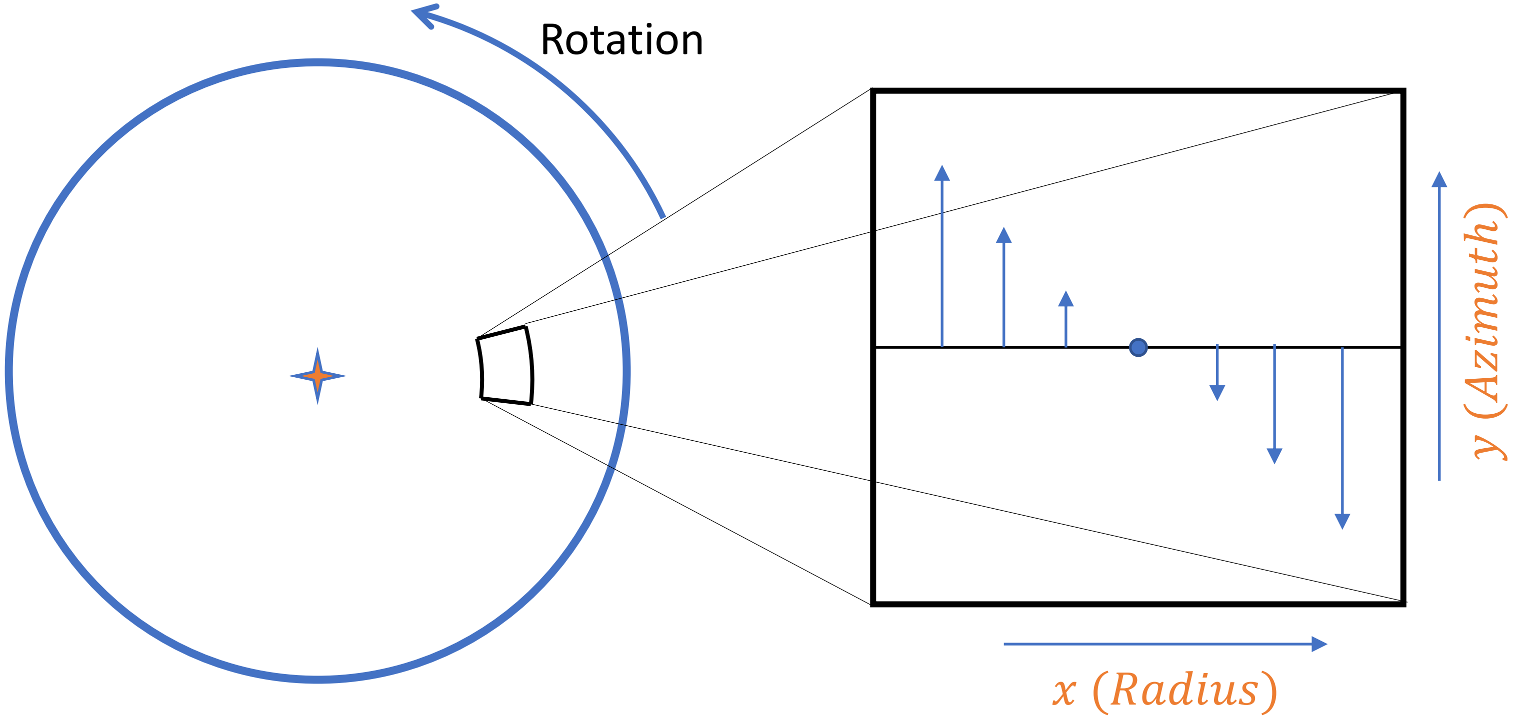


Fluid equations

$$\frac{d\Sigma}{dt} + \nabla \cdot (\Sigma \vec{v}) = 0 \quad \text{(Continuity)}$$

Origin of the Toomre Q

Shearing-box Geometry



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$$\frac{d\vec{v}}{dt} + (\vec{v} \cdot \nabla)\, \vec{v} = -\frac{\nabla P}{\Sigma} - \nabla \Phi - 2\vec{\Omega} \times \vec{v} + 3\Omega x \vec{e}_x$$

Pressure

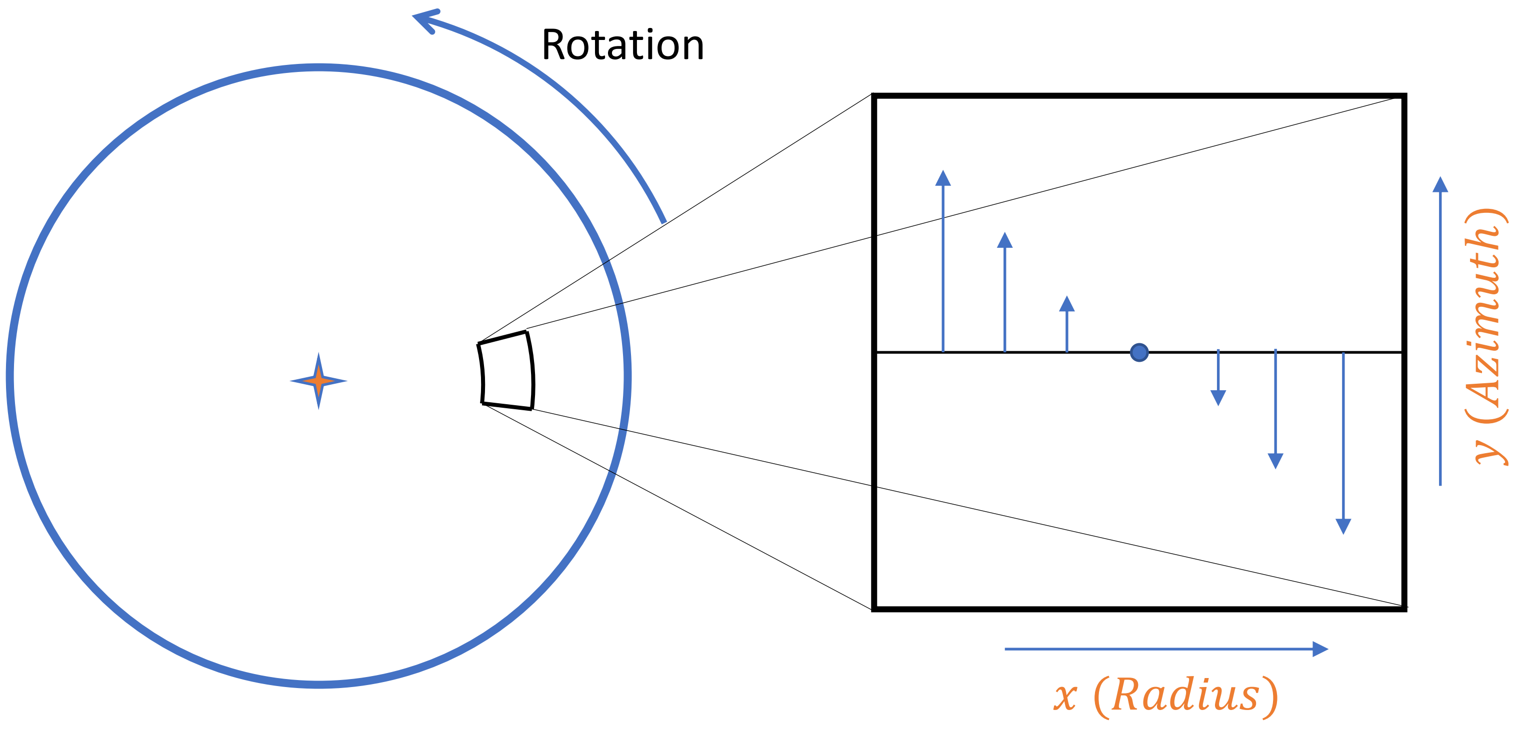
Self-gravity

Coriolis

Shear

Origin of the Toomre Q

Shearing-box Geometry



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(Momentum)

$$P = \Sigma c_s^2 \quad \text{(Pressure)} \qquad \nabla^2 \Phi = 4\pi G \Sigma \cdot \delta(z) \quad \text{(Gravity)}$$

Origin of the Toomre Q
Equilibrium solution

Fluid equation	Background solution
$\frac{d\Sigma}{dt} + \nabla \cdot (\Sigma \vec{v}) = 0$ (Continuity)	$\Sigma = \Sigma_0 = const.$
$P = \Sigma c_s^2$ (Pressure)	$c_s = const.$

Origin of the Toomre Q

Equilibrium solution

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Neglect left hand side [i.e. $(\vec{v} \cdot \nabla)\vec{v} = 0$]:

$$0 = -2\Omega v_y + 3\Omega x \quad \text{(x equation)}$$
$$0 = +2\Omega v_x \quad \text{(y equation)}$$

Origin of the Toomre Q

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$$0 = +2\Omega v_x \quad \text{(y equation)}$$

Solution:

$$v_x = 0, \qquad v_y = \frac{3}{2}\Omega x$$

(obeys $(\vec{v} \cdot \nabla)\vec{v} = 0$!)

Origin of the Toomre Q

Equilibrium solution

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Origin of the Toomre Q

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Perturb $\Sigma, \vec{v}$ and Φ:

$$\Sigma = \Sigma_0 + \tilde{\Sigma}$$
$$v_x = \tilde{v}_x$$
$$v_y = \frac{3}{2}\Omega x + \tilde{v}_y$$
$$\Phi = \tilde{\Phi}$$

Look axisymmetric solutions of the form

$$\tilde{f} = |f| \exp(st + ikx)$$

Origin of the Toomre Q

Perturb equilibrium

Fluid equation	Perturbation equation
$\frac{d\Sigma}{dt} + \nabla \cdot (\Sigma \vec{v}) = 0$ (Continuity)	$s\tilde{\Sigma} = -ik\Sigma_0 \tilde{v}_x$
$\nabla^2 \Phi = 4\pi G \Sigma \cdot \delta(z)$ (Gravity)	$\tilde{\Phi} = -\frac{2\pi G \tilde{\Sigma}}{k}$
$\frac{dv_x}{dt} + (\vec{v} \cdot \nabla) v_x = -\frac{c_s^2}{\Sigma} \frac{\mathrm{d}P}{dx} + 2\Omega v_y + 3\Omega x - \frac{\mathrm{d}\Phi}{dx}$	$s\tilde{v}_x = -\frac{c_s^2}{\Sigma_0} ik\tilde{\Sigma} + 2\Omega \, \tilde{v}_y - ik\tilde{\Phi}$
$\frac{dv_y}{dt} + (v \cdot \nabla) v_y = 2\Omega v_x$	$s\tilde{v}_y = \frac{1}{2}\Omega \, \tilde{v}_x$

Look for solutions of the form

$$\tilde{f} = |f| \exp(st + ikx)$$

Origin of the Toomre Q

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$\frac{dv_x}{dt} + (\vec{v} \cdot \nabla) v_x = -\frac{c_s^2}{\Sigma}\frac{\mathrm{d}P}{dx} - \frac{\mathrm{d}\Phi}{dx} + 2\Omega v_y + 3\Omega x$	$s\tilde{v}_x = -\frac{c_s^2}{\Sigma_0} ik\tilde{\Sigma} + 2\Omega\, \tilde{v}_y - ik\tilde{\Phi}$
$\frac{dv_y}{dt} + (v \cdot \nabla) v_y = 2\Omega v_x$	$s\tilde{v}_y = \frac{1}{2}\Omega\, \tilde{v}_x$

Look for solutions of the form

$$\tilde{f} = |f| \exp(st + ikx)$$

Next:

- Replace, $\tilde{\Sigma}$, $\tilde{\Phi}$, and $\tilde{v}_y$ with expressions for $\tilde{v}_x$

Origin of the Toomre Q

Perturbed equation for $\tilde{v}_x$

$$s\tilde{v}_x = -\frac{c_s^2}{\Sigma_0} ik\tilde{\Sigma} + 2\Omega\,\tilde{v}_y - ik\tilde{\Phi}$$

$$s^2\tilde{v}_x = -c_s^2\,\boldsymbol{k}^2\tilde{v}_x - \Omega^2\,\tilde{v}_x + 2\pi G\Sigma_0\Omega\,\boldsymbol{k}\tilde{v}_x$$

Pressure Coriolis/Shear — Stabilizing

Self-gravity — De-stabilizing

Perturbation equation
$s\tilde{\Sigma} = -ik\Sigma_0\tilde{v}_x$
$\tilde{\Phi} = -\frac{2\pi G\tilde{\Sigma}}{k}$
$s\tilde{v}_x = -\frac{c_s^2}{\Sigma_0} ik\tilde{\Sigma} + 2\Omega\,\tilde{v}_y - ik\tilde{\Phi}$
$s\tilde{v}_y = \frac{1}{2}\Omega\,\tilde{v}_x$

Origin of the Toomre Q

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$$s\tilde{v}_x = -\frac{c_s^2}{\Sigma_0} ik\tilde{\Sigma} + 2\Omega\,\tilde{v}_y - ik\tilde{\Phi}$$

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Instability exists when $s^2 > 0$.

From the quadratic equation:

$$ax^2 + bx + c > 0$$

we know solutions exist for $b^2 > 4ac$.

Perturbation equation
$s\tilde{\Sigma} = -ik\Sigma_0\tilde{v}_x$
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Origin of the Toomre Q

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Instability exists when $s^2 > 0$.

From the quadratic equation:

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we know solutions exist for $b^2 > 4ac$.

Hence:

$$4\,(\pi G\Sigma_0)^2 > 4\, c_s^2\Omega^2 \quad \text{or} \quad \frac{1}{Q^2} > 1$$

Perturbation equation
$s\tilde{\Sigma} = -ik\Sigma_0\tilde{v}_x$
$\tilde{\Phi} = -\frac{2\pi G\tilde{\Sigma}}{k}$
$s\tilde{v}_x = -\frac{c_s^2}{\Sigma_0} ik\tilde{\Sigma} + 2\Omega\, \tilde{v}_y - ik\tilde{\Phi}$
$s\tilde{v}_y = \frac{1}{2}\Omega\, \tilde{v}_x$

Q^2 is the ratio of the two stabilizing forces to the square of the destabilizing forces.

Origin of the Toomre Q

Fastest growing modes

$$s^2 = -c_s^2 \; k^2 - \Omega^2 \; \tilde{v}_x + 2\pi G \Sigma_0 \Omega \; k$$

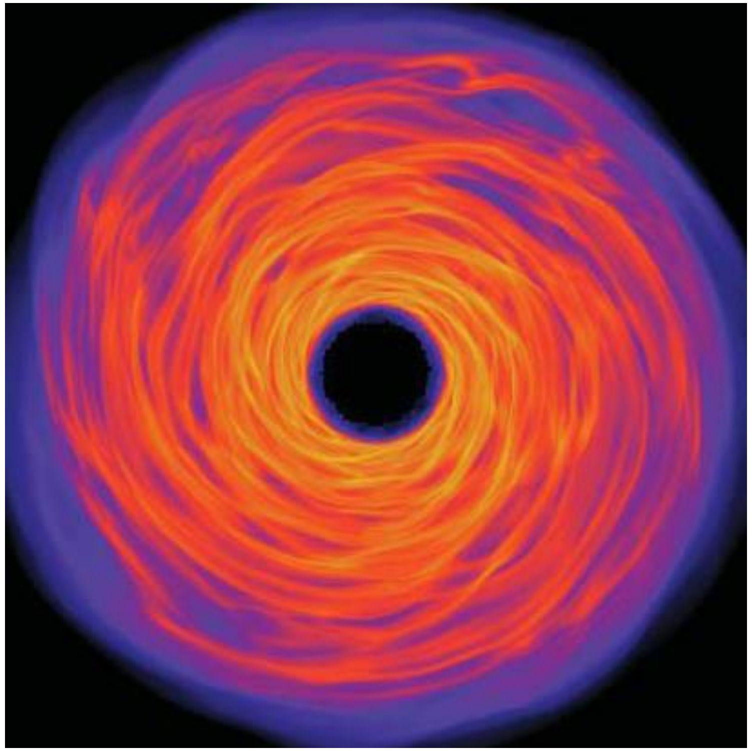
Write $H = c_s/\Omega$ and rescale:

$$\left(\frac{s}{\Omega}\right)^2 = -(kH)^2 - 1 + \frac{2}{Q} \; (kH)$$

Fastest growing mode has $\frac{ds}{d(kH)} = 0.$

$$\rightarrow (kH)_{\max} = 1/Q$$

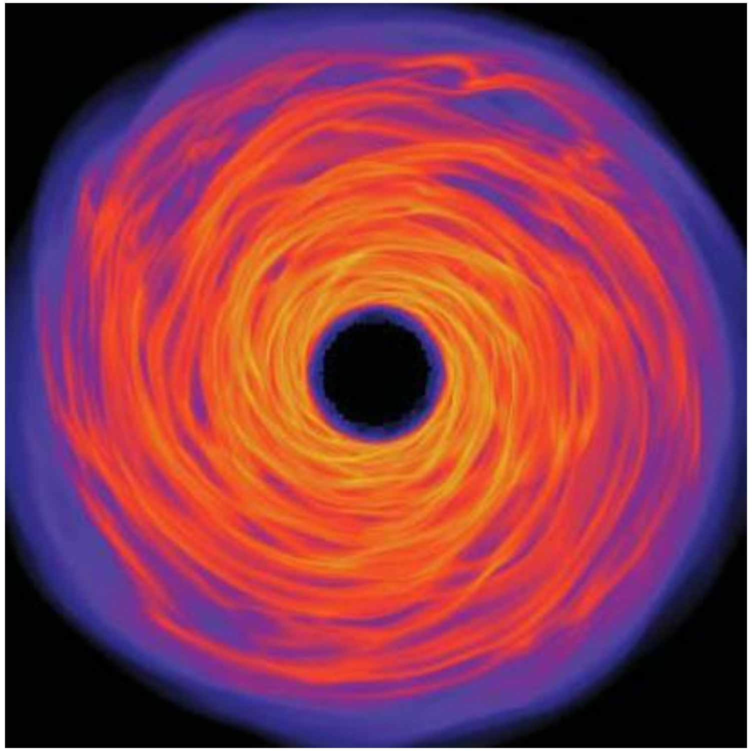
$$\left(\frac{s}{\Omega}\right)^2_{\max} = \frac{1}{Q^2} - 1$$



Outcome of Gravitational Instability

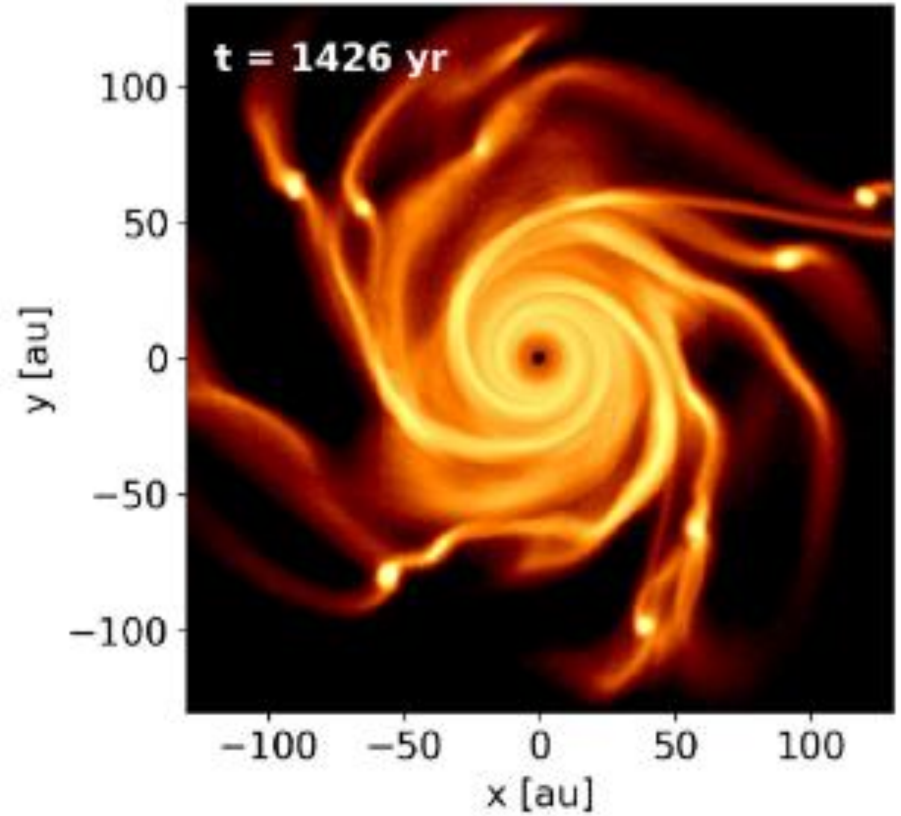
Outcome of Gravitational Instability

Quasi-steady accretion



Simulation from Booth & Clarke (2016)

Fragmentation



Simulation from Ilee et al. (2017)

Cooling dictates the outcome:

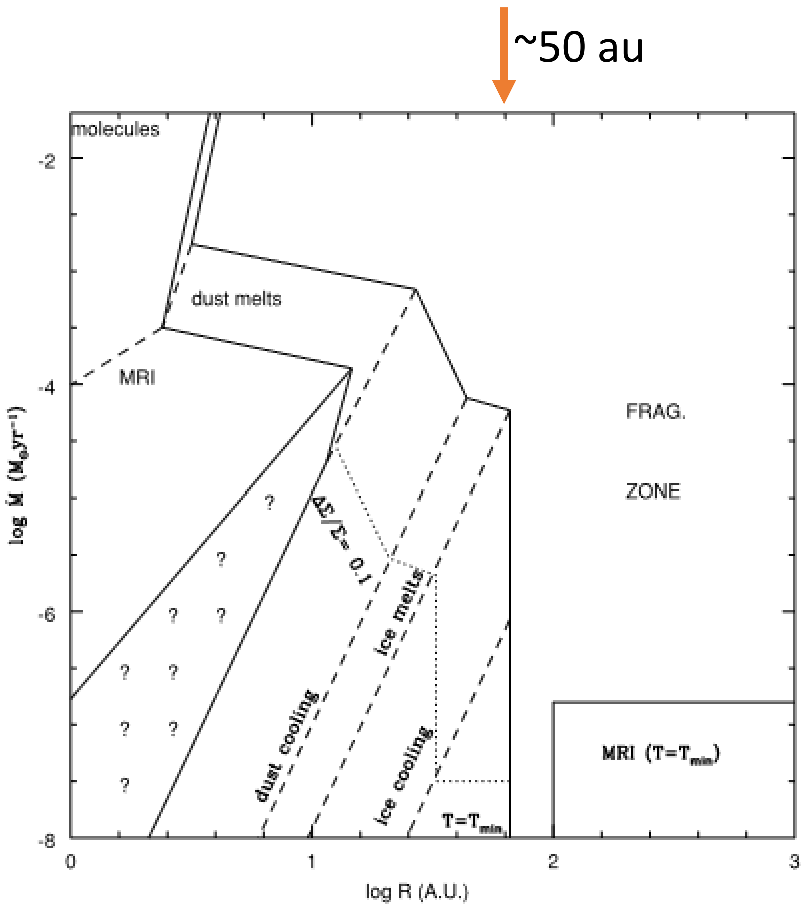
Fast cooling ($\tau_{cool} < 3\Omega^{-1}$):

➔ fragmentation

Outcome of Gravitational Instability

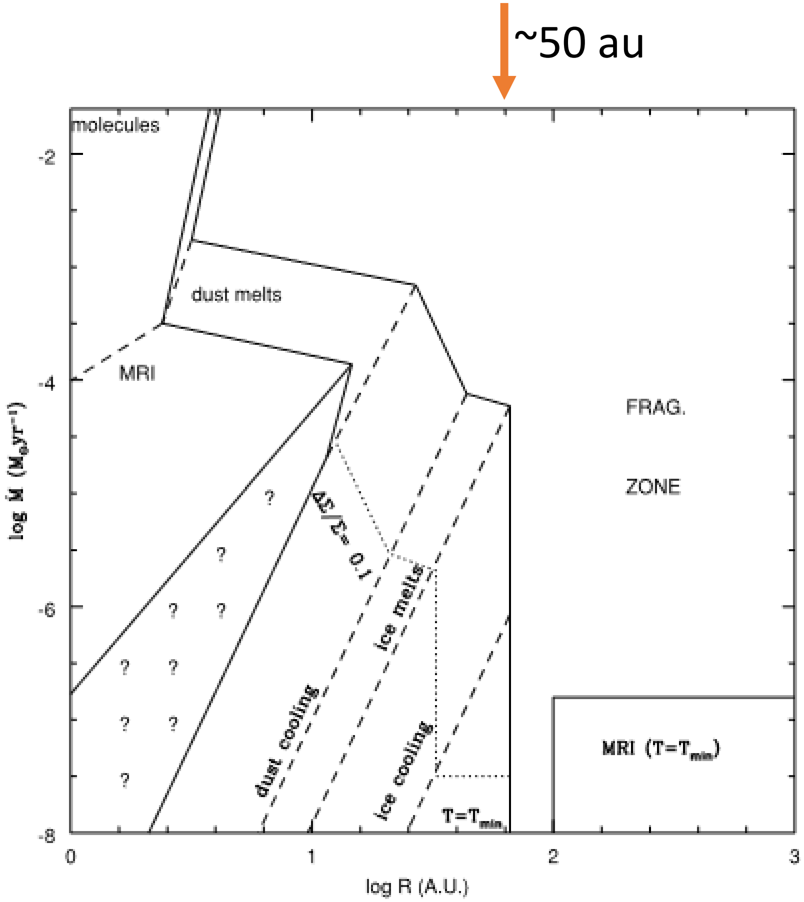
Fragmentation

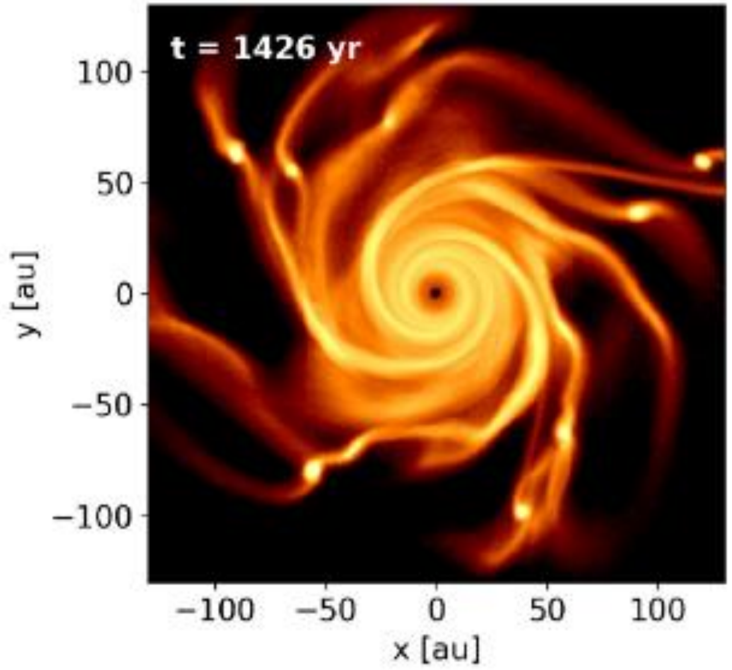
Where do discs fragment?



Clarke & Lodato (2009)

Where do discs fragment?





Simulation from Ilee et al. (2017)

Clarke & Lodato (2009)

Does fragmentation produce planets?

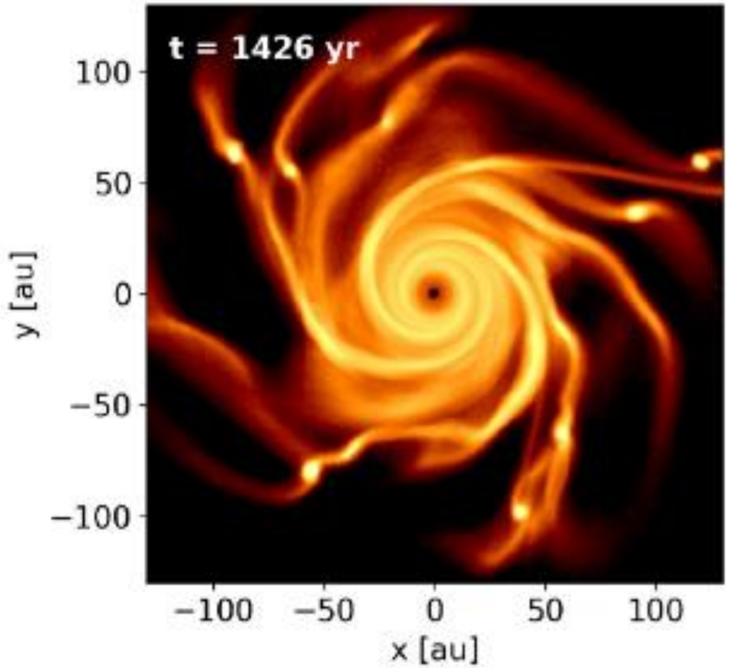
- Initial fragment masses:
  - Set by size of the collapsing region
    - ~Wavelength of most unstable mode

$$M_{frag} \sim \Sigma\,\lambda^2$$

Since $kH = \frac{1}{Q}$ and $Q = \frac{c_s \Omega}{\pi G \Sigma} = \frac{H\Omega^2}{\pi G \Sigma}$:

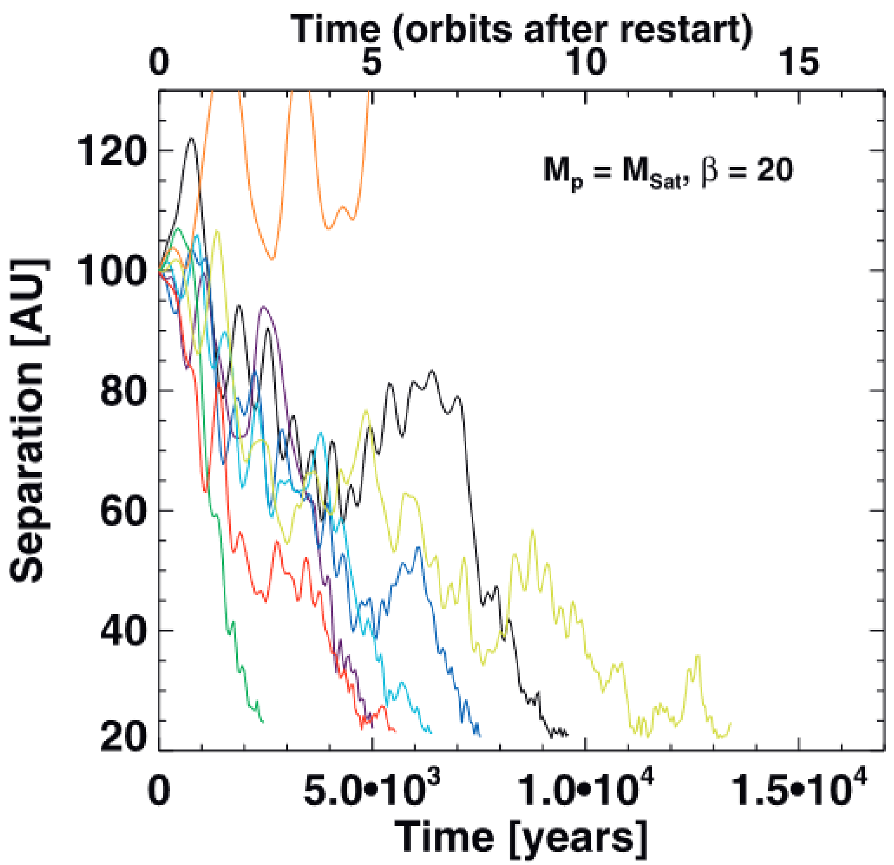
$$M_{frag} \sim 4\pi Q \left(\frac{H}{R}\right)^3 M_* \approx 10 M_J$$

Fragments are super-Jupiter or Brown Dwarf mass objects



Fate of the fragments

Fast migration

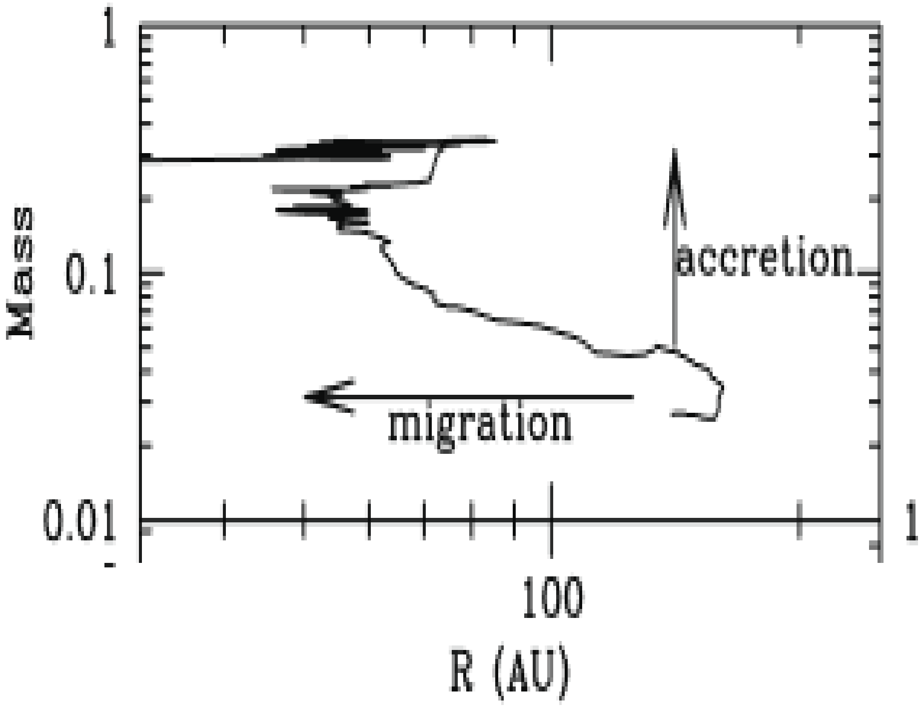


- Saturn-mass planet
- Similar results for all planet masses

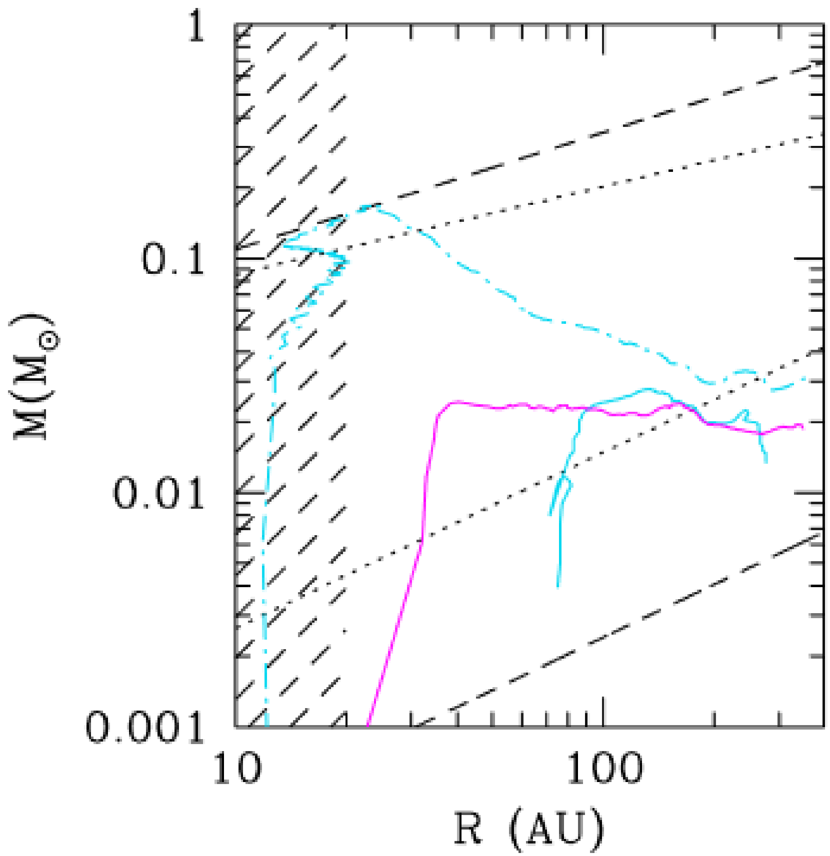
Baruteau+ (2011)

Fate of the fragments
Disruption, or high masses

Strong accretion



Tidal disruption



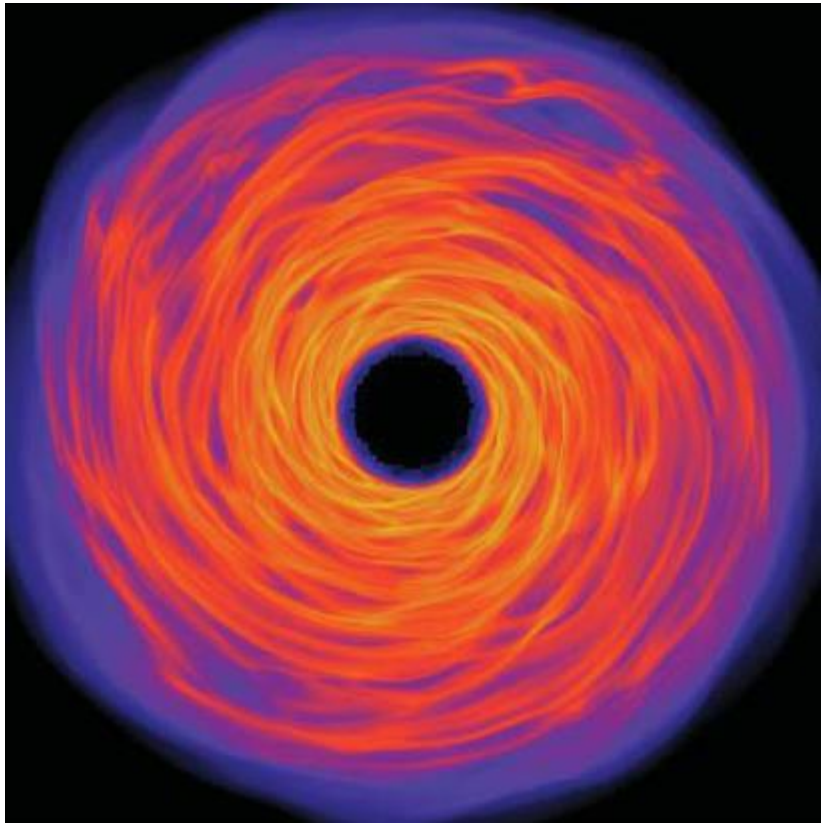
Giant planets are likely a rare outcome of GI

Zhu+ (2012)

Outcome of Gravitational Instability

Quasi-steady gravito-turbulence

Why is there a quasi-steady state?



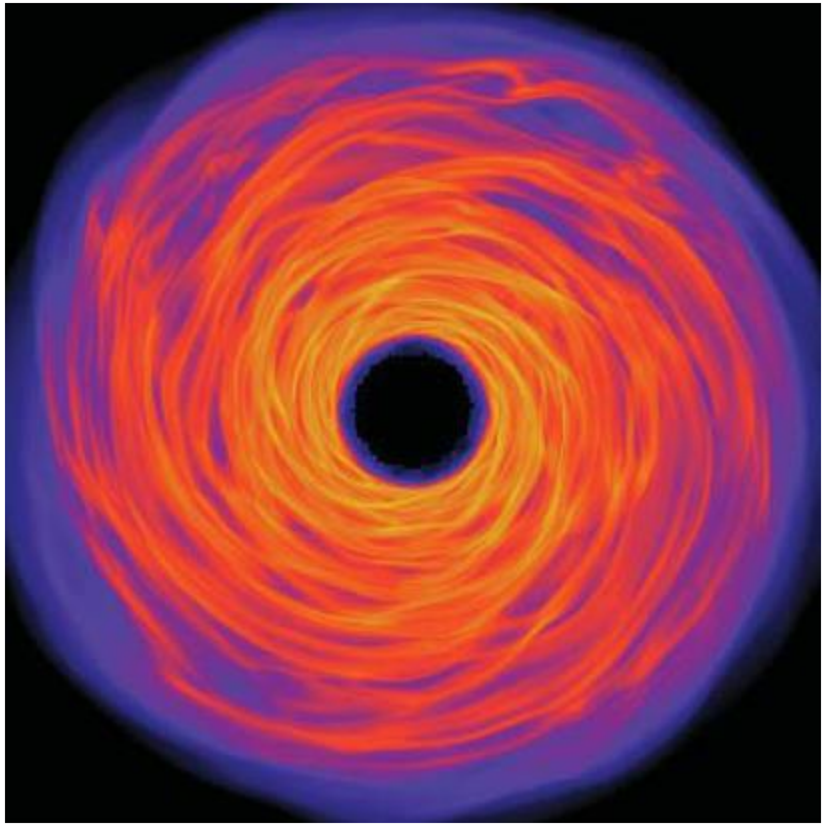
Why is there a quasi-steady state?

Let's look at the growth rate of GI:

$$\left(\frac{s}{\Omega}\right)^2 = -(kH)^2 - 1 + \frac{2}{Q}\,(kH)$$

With the fastest growing mode:

$$\left(\frac{s}{\Omega}\right)^2_{\max} = \frac{1}{\mathrm{Q}^2} - 1$$



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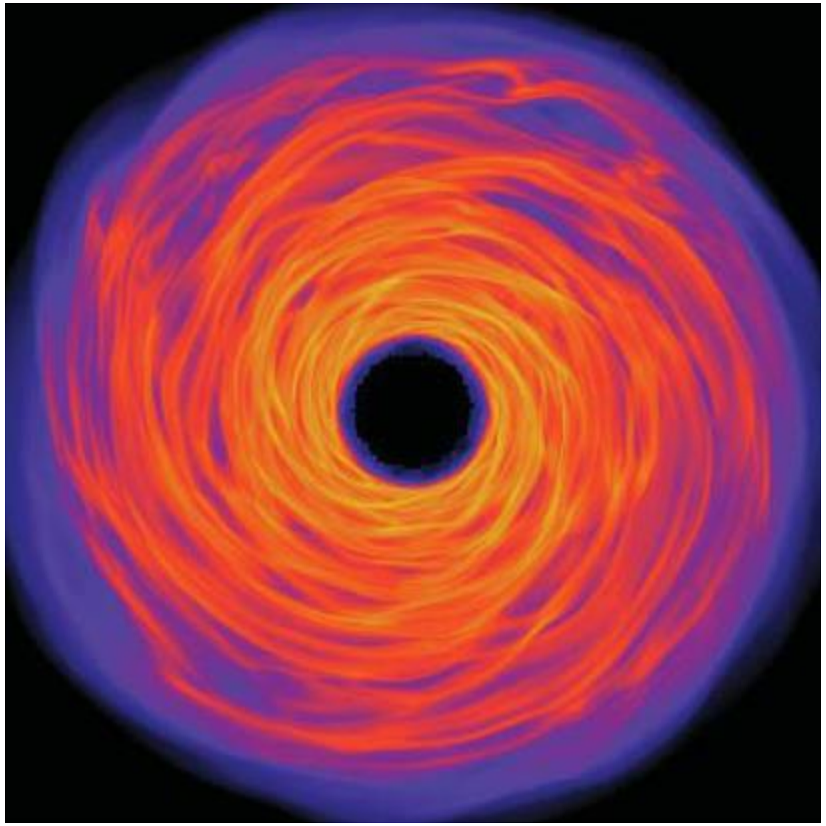
Cooling brings Q *just* low enough that GI grows *just* fast enough to replenish the lost energy



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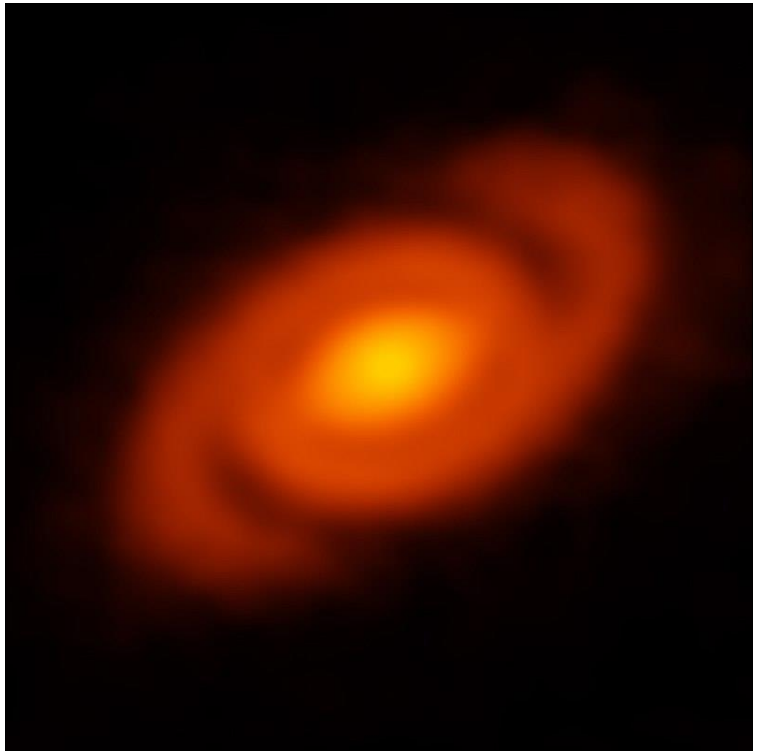
Isothermal simulations can't self-regulate

➔ *they are much more prone to fragmentation*



Is Elias 27 a post-fragmentation system? (Perez et al. 2016)

Local nature of gravito-turbulence

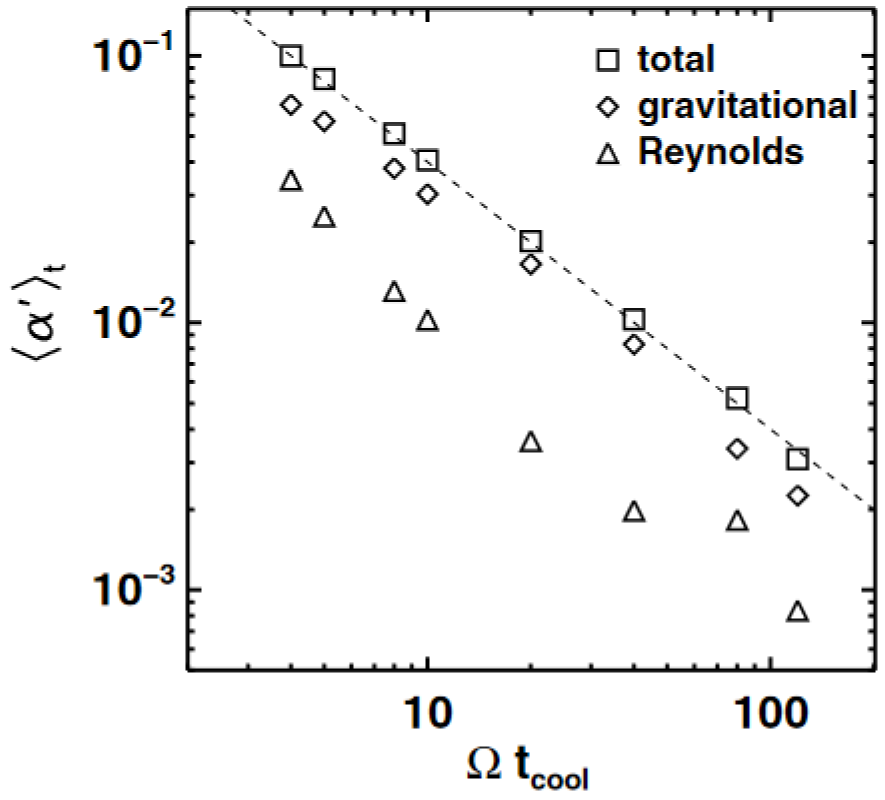
Effective viscosity

- Q and s depend on the cooling rate
  - ➔Physical properties depend on cooling

Most famous is the effective viscosity:

$$\alpha = \frac{4}{9\gamma(\gamma-1)} \frac{1}{\tau_{cool}\Omega}$$ (Gammie 2001)

- Requires heating/cooling to occur in the same place



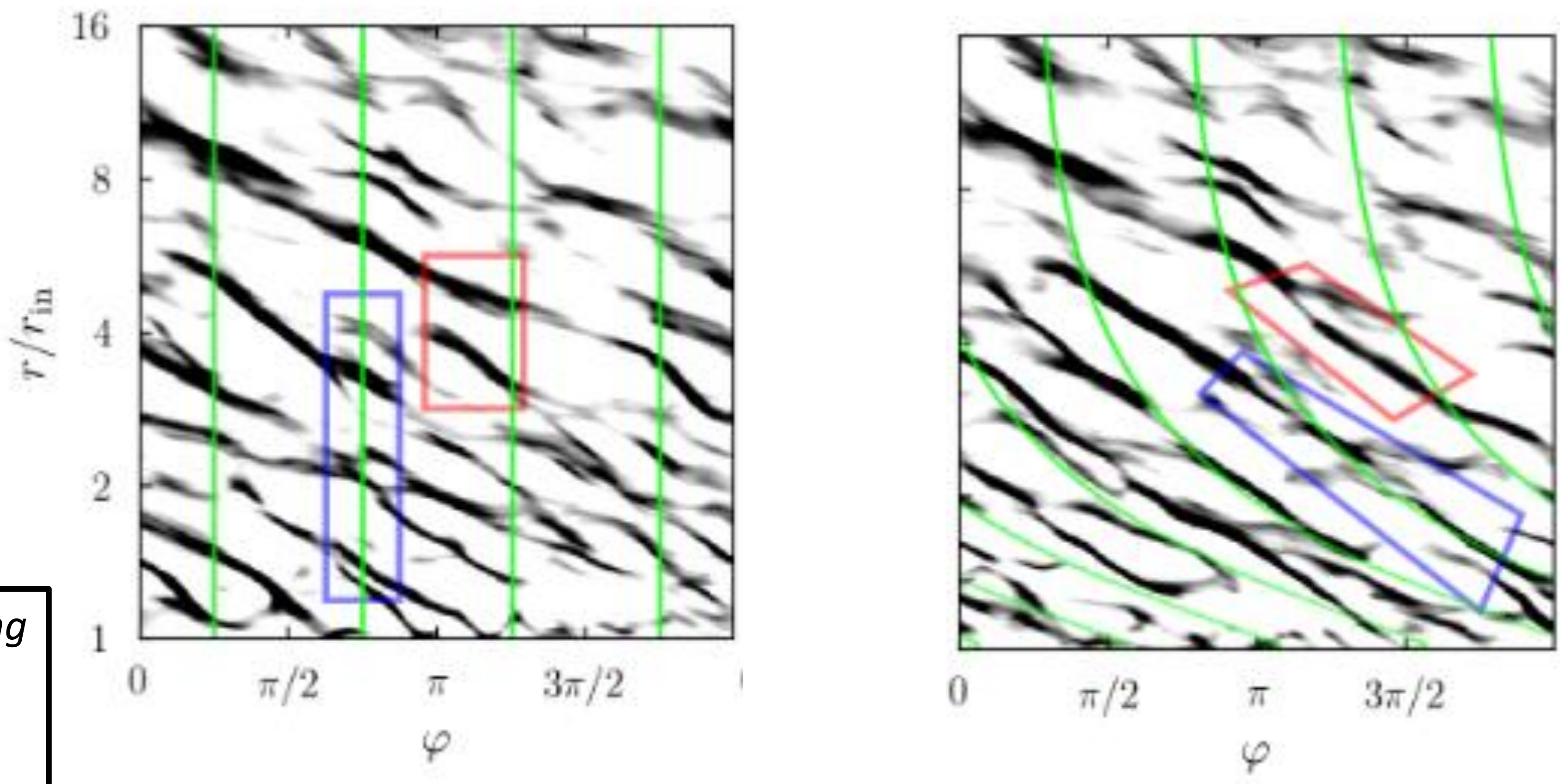
(Shi & Chiang 2015)

Local nature of gravito-turbulence
Spiral structure at two times

Green:
- Show Keplerian motion

Red/Blue:
- Demonstrate formation + destruction of large features

Gravito-turbulence is not travelling waves, but features that appear and disappear on orbital time-scales



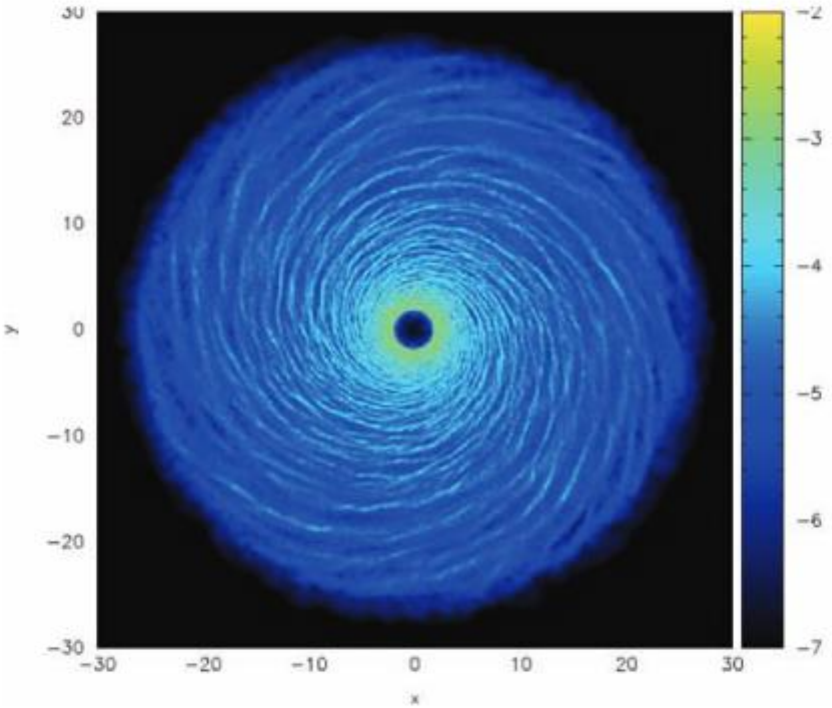
Bethune + (2021)

Structure depends on disc mass

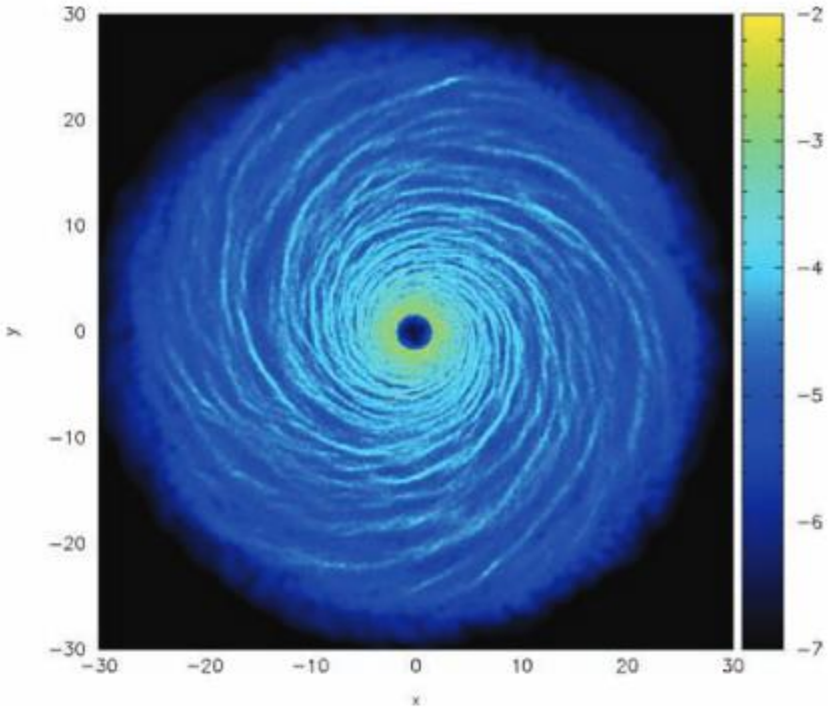
Number of spiral arms depends on disc mass.

Why?

Less Massive



More Massive



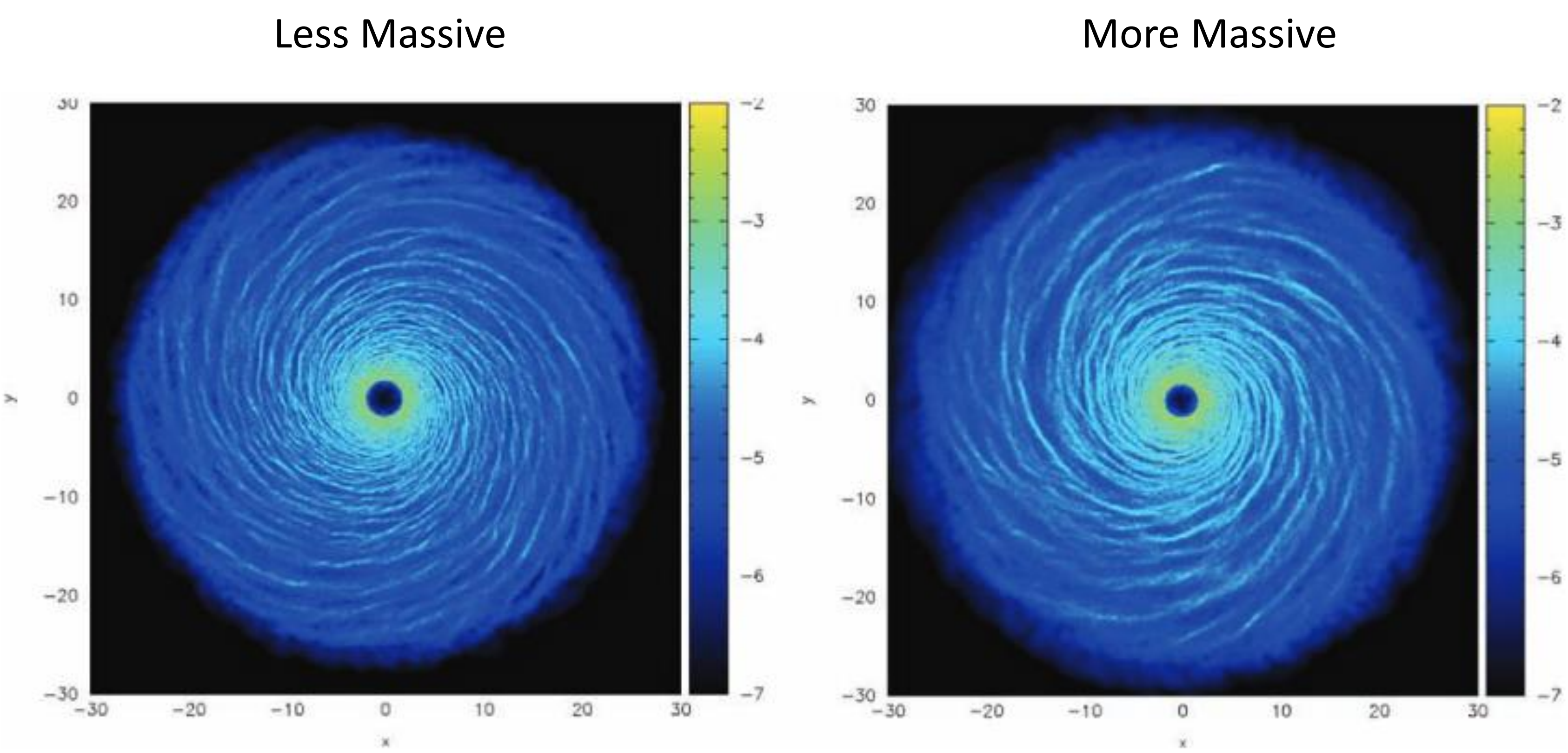
Cossins+ (2009)

Dependence on disc mass

Number of spiral arms depends on disc mass.

Why?

$$c_s = Q\frac{\pi G \Sigma}{\Omega}$$



Cossins+ (2009)

Dependence on disc mass

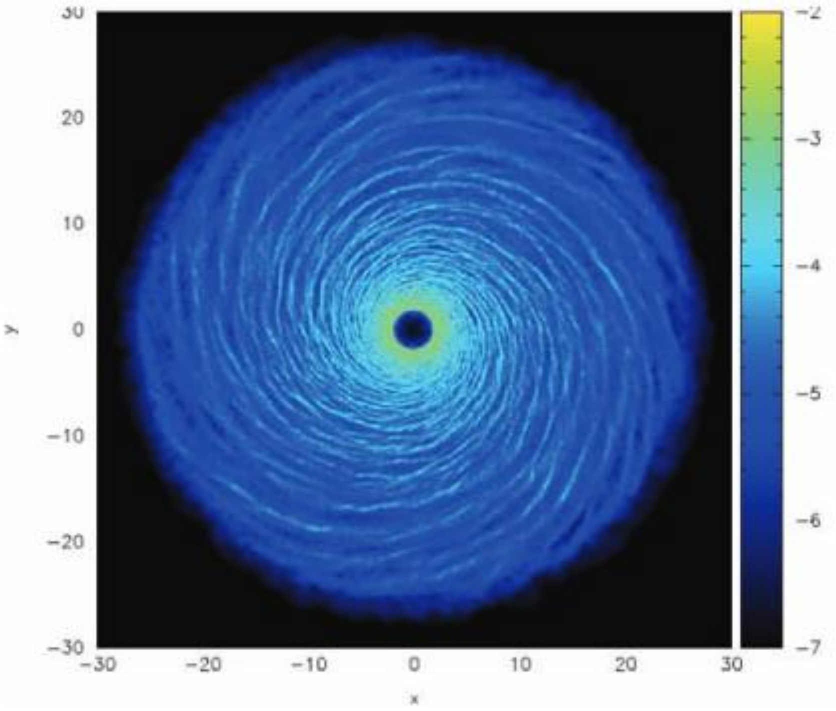
Number of spiral arms depends on disc mass.

Why?

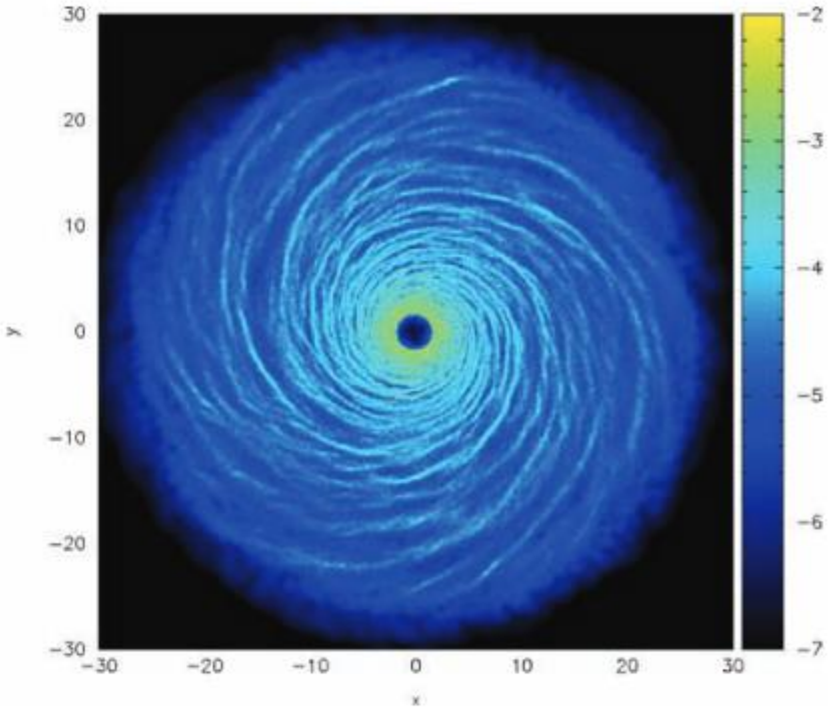
$$c_s = Q\frac{\pi G \Sigma}{\Omega}$$

$$H = Q\frac{\pi G \Sigma}{\Omega^2} = Q\frac{\pi G \Sigma R^3}{G M_*}$$

Less Massive



More Massive



Cossins+ (2009)

Dependence on disc mass

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Why?

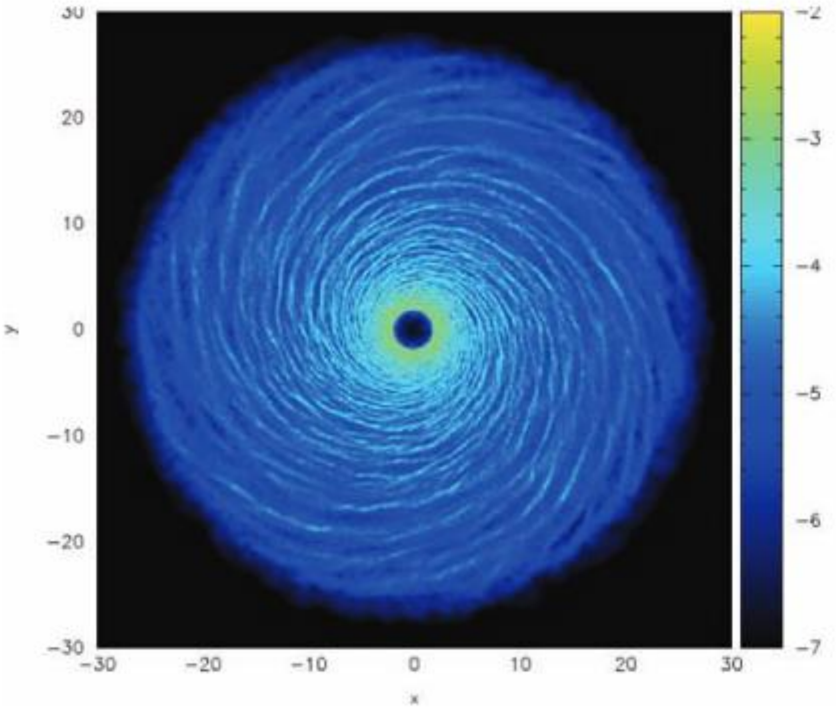
$$c_s = Q\frac{\pi G\Sigma}{\Omega}$$

$$H = Q\frac{\pi G\Sigma}{\Omega^2} = Q\frac{\pi G\Sigma R^3}{GM_*}$$

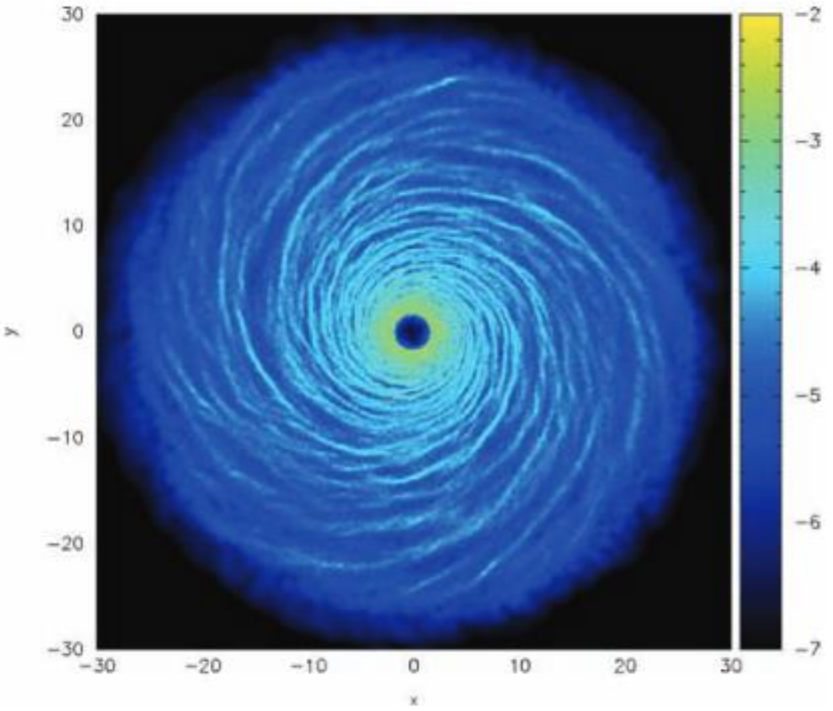
Or

$$\frac{H}{R} = Q\frac{\pi R^2\Sigma}{M_*} \sim Q\frac{M_d}{M_*}$$

Less Massive



More Massive



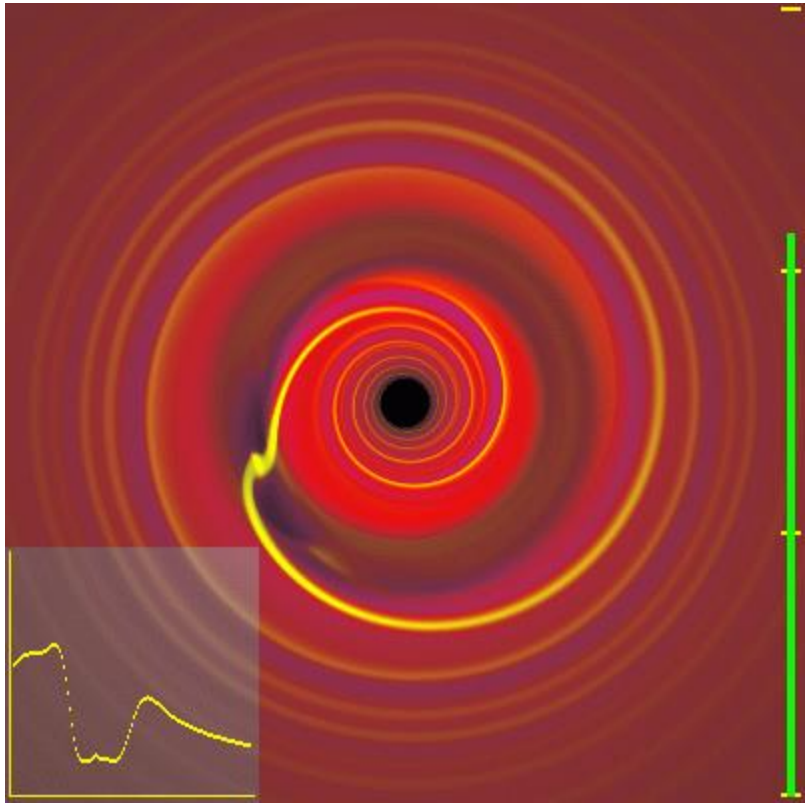
Cossins+ (2009)

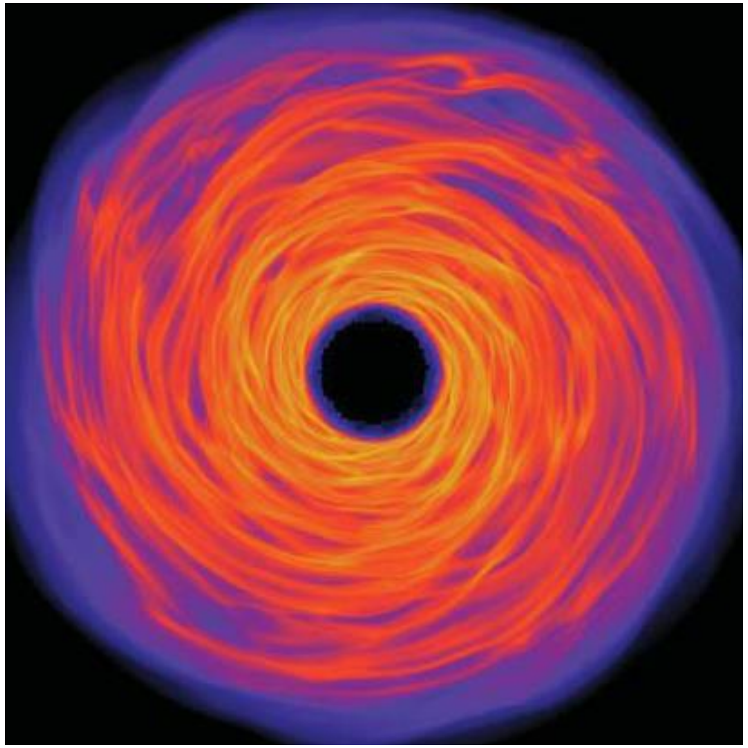
Most unstable wavelength scales with H.

➔ Number of spirals $\sim \frac{H}{R} \sim \frac{M_d}{M_*}$

Project:
Dust in quasi-steady GI discs

Planet vs GI spirals



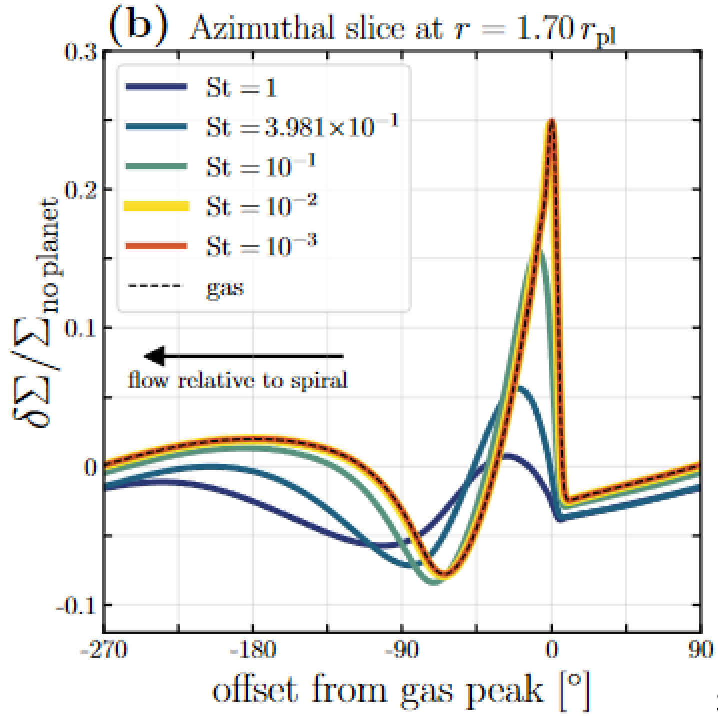


- Spiral rotates with planet
- Long lived steady spiral

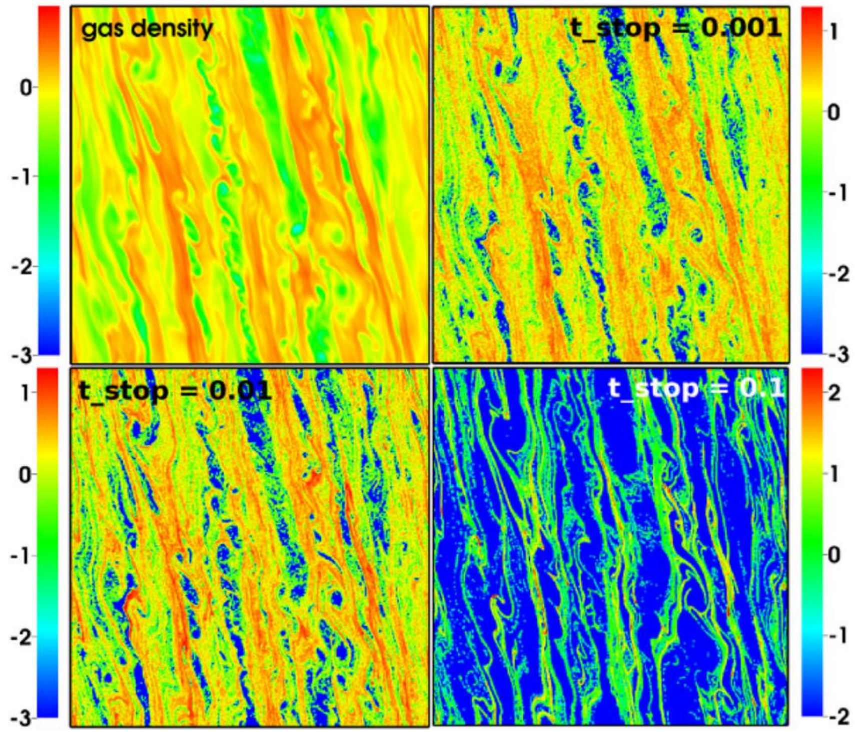
- Spiral rotates with the gas
- Individual spirals are short lived

Dusty spirals
Differences in structure

Problem:
Can we tell planet and GI induced spirals apart observationally by their grain-size dependence?



Speedie et al. (2022)



- Strongest for small grains
- Strongest for large grains

Shi et al. (2016)

Dusty Spirals

GI

Approach:

- Use FARGO3D to simulate dust dynamics in spiral discs
- Explore how the dust-to-gas ratio varies for different sizes
- Compute simulated observations
- Compare with planet case

Key questions:

- What disc parameters should you choose?
- What dust parameters?
- What else might matter?