Dustbusters summer school
16th May–20th May 2022

A crash course in astrophysical fluid dynamics

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You can view the online lectures associated with this material at:
https://www.youtube.com/watch?v=MOjPtBFGtSI&list=PLMzuj51UjsPTZjHd6XKB4PYbqYDsEBKwH&index=5

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1 Compressible hydrodynamics

Before we solve anything on the computer we need to understand what it is we're trying to do. I will therefore start with a quick introduction to compressible hydrodynamics. It will be a whirlwind tour because we want to focus on the computational aspects in this course. But first we need to go over the basic maths and physics of the fluid equations in order to understand our computational approach.

Let's start with compressible hydrodynamics. What do we even mean by "hydrodynamics"? We mean a particular set of partial differential equations — the equations of hydrodynamics. The inviscid version of these equations are called the Euler equations, after Leonard Euler (1707–1783). If we add viscosity these would be called the Navier-Stokes equations. But the Navier Stokes equations are usually applied to incompressible flow, whereas in astrophysical fluid dynamics we deal almost entirely with compressible and nearly inviscid flow.

1.1 Equations of hydrodynamics

The fluid equations[1] are simply mathematical expressions of physical rules. These rules are simple: mass, momentum and energy must be conserved!

1.1.1 Conservation of mass

Conservation of mass is expressed by the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{1}$$

where ρ is the density and $\boldsymbol{v}$ is the velocity (a vector) and $\nabla \cdot$ is the vector calculus divergence operator[2], here acting on the vector $\rho \boldsymbol{v}$. Equation (1) expresses the "rule" that mass must be conserved. Expanding the second term, we can also write this equation in the form

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho = -\rho(\nabla \cdot \boldsymbol{v}). \tag{2}$$

[1] When we refer to 'fluids', this could be solid, liquid or gas. In this course we are mostly concerned with gas because there are not too many liquids in space.

[2] See vector calculus revision notes if you are not confident working in vector notation

1.1.2 Conservation of momentum

Our next rule is conservation of momentum. It is expressed by the 'momentum equation'

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho} + \boldsymbol{a}_{\rm ext}, \tag{3}$$

where P is the pressure and $\boldsymbol{a}_{\rm ext}$ refers to any external acceleration (e.g. gravity) applied to the fluid. The main term of interest that accelerates the fluid is the $-\nabla P/\rho$ term, namely the acceleration caused by gradients (i.e. differences) in pressure.

1.1.3 Conservation of energy

Finally, we have conservation of energy. We are going to express this by writing down an equation for the internal energy per unit mass, u, given by

$$\frac{\partial u}{\partial t} + (\boldsymbol{v} \cdot \nabla)u = -\frac{P}{\rho}(\nabla \cdot \boldsymbol{v}) + \Lambda_{\rm heat} - \Lambda_{\rm cool}, \tag{4}$$

where $\Lambda_{\rm heat}$ and $\Lambda_{\rm cool}$ represent any external heating and cooling applied to the fluid. As we will see later, one can write down equivalent equations expressing conservation of energy in terms of other variables, e.g. the total specific energy $e \equiv \frac{1}{2}v^2 + u$ instead of u.

1.1.4 Summary of fluid equations

In summary, the fluid equations expressing conservation of mass, momentum and energy are given by

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho = -\rho(\nabla \cdot \boldsymbol{v}), \tag{5}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho} + \boldsymbol{a}_{\rm ext}, \tag{6}$$

$$\frac{\partial u}{\partial t} + (\boldsymbol{v} \cdot \nabla)u = -\frac{P}{\rho}(\nabla \cdot \boldsymbol{v}) + \Lambda_{\rm heat} - \Lambda_{\rm cool}. \tag{7}$$

Here all of our derivatives are written as partial, or *Eulerian* derivatives.

1.1.5 Advection and the Lagrangian time derivative

You may notice already that the left hand side of our equations look similar. This is no accident — the '$\boldsymbol{v} \cdot \nabla$' terms simply express the fact that the fluid is moving and that the fluid properties (ρ, $\boldsymbol{v}$ and u) are properties that are 'carried' or *advected* by the flow.

We can simplify these expressions by defining the 'co-moving' or 'Lagrangian' time derivative. We will use $\mathrm{d}/\mathrm{d}t$ to distinguish this operator from the *Eulerian* time derivative $\partial/\partial t$. We define the *Lagrangian* time derivative according to

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \tag{8}$$

So for example, applying this derivative to the density we would have

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho. \tag{9}$$

The way to understand the Lagrangian derivative is as the time derivative of a co-moving observer. What does this mean? An easy way to think about the Lagrangian derivative is as follows: It is the operator such that the time derivative of the position $\boldsymbol{x} \equiv [x, y, z]$ equals the velocity, i.e.

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}. \tag{10}$$

In particular, it does not make sense to use the partial/Eulerian time derivative here, since $\partial \boldsymbol{x}/\partial t \neq \boldsymbol{v}$.

Lagrangian and Eulerian observers

The Lagrangian derivative corresponds to the time derivative seen by an observer co-moving with the flow. Imagine you have a river, and a flotilla of equally-spaced boats are floating along the river at some (constant) speed $\boldsymbol{v}_{\mathrm{river}}$ and one wishes to know how the density of boats, ρ_{b}, changes with time. For an observer sitting on one of the boats (a *Lagrangian*, or *co-moving* observer), the local density of boats is constant, so $\mathrm{d}\rho_{\mathrm{b}}/\mathrm{d}t = 0$. However for an observer sitting on the bank watching the flotilla pass, the density is zero, rises to a constant and then fades to zero with time as the flotilla passes. This *Eulerian* observer sees $\partial \rho_{\mathrm{b}}/\partial t \neq 0$ — more precisely, $\partial \rho_{\mathrm{b}}/\partial t = -\boldsymbol{v}_{\mathrm{river}} \cdot \nabla \rho_{\mathrm{b}}$.

1.1.6 Equations of hydrodynamics in Lagrangian form

With the aid of the Lagrangian derivative, our fluid equations simplify to

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho(\nabla \cdot \boldsymbol{v}), \tag{11}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla P}{\rho} + \boldsymbol{a}_{\rm ext}, \tag{12}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{P}{\rho}(\nabla \cdot \boldsymbol{v}) + \Lambda_{\rm heat} - \Lambda_{\rm cool}. \tag{13}$$

Much simpler! For the remainder of these notes we will assume no external forces ($\boldsymbol{a}_{\rm ext} = 0$) and no heating or cooling ($\Lambda_{\rm heat} = \Lambda_{\rm cool} = 0$).

1.2 Equation of state

We have 5 equations, evolving ρ, $\boldsymbol{v}$ and u, but 6 unknowns (ρ, $\boldsymbol{v}$, u and P). So the equations cannot be solved unless we provide an additional equation to act as a *closure relation*. This should be an equation that connects the remaining unknown P to known quantities, namely ρ and u. This is known as the *equation of state*.

1.2.1 Ideal gas

For an ideal gas we have $P = nk_{\rm B}T$, where n is the number density. Expressing this in terms of the mass density ρ gives

$$P = \frac{\rho k_{\rm B} T}{\mu m_u}, \tag{14}$$

where $k_{\rm B}$ is the Boltzmann constant, T is the temperature, μ is the mean molecular weight (e.g. $\mu = 2$ for a gas made of molecular Hydrogen) and m_u is the atomic mass unit (i.e. mass of a proton, or more precisely 1/12 the mass of a Carbon-12 atom)[3].

However, the fluid motion is not influenced by the temperature or the composition, only by the pressure. Thus it is more helpful to express the equation of state in terms of the internal energy, u, instead of T. We write

$$P = (\gamma - 1)\rho u, \tag{15}$$

[3] I personally avoid use of the gas constant $\mathcal{R}$ in astrophysics since i) there are two different definitions and ii) we rarely need to deal with moles.

where γ is the *ratio of specific heats*. Comparing (14) and (15), we find

$$u \equiv \frac{1}{(\gamma - 1)} \frac{k_{\rm B} T}{\mu m_u}. \tag{16}$$

From this equation we see that using the internal energy means we don't need to worry about the composition. That is, we only need to supply μ if we wanted to interpret our results in terms of temperature. The value of γ depends on the number of degrees of freedom in the gas. For a monatomic gas [a gas where the particles are single atoms], we have

$$u = \frac{3}{2} n k_{\rm B} T = \frac{3}{2} \frac{k_{\rm B} T}{\mu m_u}, \tag{17}$$

so we deduce that $\gamma = 5/3$ for this case.

1.2.2 Solving the energy equation with no external heating or cooling

A simpler equation of state can be employed in the case where there is no external heating or cooling. In this case our energy equation (13) becomes simply

$$\frac{{\rm d}u}{{\rm d}t} = -\frac{P}{\rho}(\nabla \cdot \boldsymbol{v}) = \frac{P}{\rho^2} \frac{{\rm d}\rho}{{\rm d}t}. \tag{18}$$

where in the last step we substituted the continuity equation in the form (11). If we then assume the equation of state in the form (15) we have

$$\frac{1}{u} \frac{{\rm d}u}{{\rm d}t} = \frac{(\gamma - 1)}{\rho} \frac{{\rm d}\rho}{{\rm d}t}, \tag{19}$$

giving

$$\frac{{\rm d} \ln u}{{\rm d}t} = (\gamma - 1) \frac{{\rm d} \ln \rho}{{\rm d}t} = \frac{{\rm d} \ln \rho^{(\gamma - 1)}}{{\rm d}t}. \tag{20}$$

Dropping the dt and integrating both sides we have

$$\int {\rm d} \ln u = \int {\rm d} \ln \rho^{(\gamma - 1)}, \tag{21}$$

$$\therefore \ln u = \ln \rho^{(\gamma - 1)} + C, \tag{22}$$

$$\therefore u = \tilde{K} \rho^{(\gamma - 1)}, \tag{23}$$

giving, using (15)

$$P = K\rho^{\gamma}, \tag{24}$$

where C and $\tilde{K}$ and K are arbitrary constants. Equation (24) is referred to as a *polytropic* equation of state and is the exact solution to the energy equation for an adiabatic gas with no irreversible heating or cooling. In this case we do not need to solve (13) on the computer, one can simply use the analytic solution (24) directly.

1.2.3 Isentropic gas

We can also consider (24) as referring to an *isentropic* (constant entropy) gas because from the first law of thermodynamics we have

$$T\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}t} - \frac{P}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t}, \tag{25}$$

which equals zero if the gas is adiabatic, since $\mathrm{d}u/\mathrm{d}t = P/\rho^2 \mathrm{d}\rho/\mathrm{d}t$. Now consider the Lagrangian time derivative of the quantity $K = P/\rho^{\gamma}$, which gives

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P}{\rho^{\gamma}}\right) = \frac{\gamma - 1}{\rho^{\gamma-1}}\left(\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{P}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t}\right). \tag{26}$$

So assuming $K = \text{const}$ is equivalent to stating that there is no change in entropy.

1.2.4 Isothermal gas

In the case where $\gamma = 1$ in (24) we obtain $P \propto \rho$. From our ideal gas law (14) we can see that this corresponds to $T = \text{const.}$. So we refer to this as an *isothermal* — constant temperature — equation of state. It is convenient to write (24) in the form

$$P = c_{\mathrm{s}}^2 \rho, \tag{27}$$

where c_{s} is the (constant) isothermal sound speed. Why we identify this constant with the sound speed will become clear in Section 1.4.4. For numerical work this is the simplest possible closure — one simply needs to specify the value of c_{s} to close the equation set.

Isothermal gas as a singular limit

The limit of $\gamma \to 1$ is known as a *singular limit* in that analytic solutions for $\gamma \neq 1$ do not simply reduce to the $\gamma = 1$ case. One can observe this in our equations (15) and (26) where taking $\gamma \to 1$ gives nonsense. Despite this our result (24) gives the correct limit when $\gamma \to 1$. The other famous singular limit in the fluid equations is that solutions to the fluid equations *with* viscosity — the Navier Stokes equations — do not simply reduce to solutions of the fluid equations *without* viscosity — the Euler equations — as the viscosity tends to zero.

1.3 Classification of partial differential equations

Partial differential equations are classified into three different kinds: elliptic, parabolic or hyperbolic. They can also be a mix of all three. While there is a mathematical definition, we are more interested in the physics of the equations. Briefly:

Elliptic. Example: Poisson's equation for the gravitational field

$$\nabla^2 \Phi = 4\pi G \rho. \tag{28}$$

The key ingredient is that there is no time in the equation. This implies *instant action*. That is, since there is no time involved, elliptic equations require information to be propagated instantaneously across the entire computational domain. For example, if the density changes, the gravitational potential Φ must also change. Everywhere in space. Instantly. This is expensive since it requires some form of global communication in the computational domain.

Parabolic. Example: The heat equation

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u. \tag{29}$$

Here there is a single time derivative and the physics corresponds to that of *diffusion*. Dimensional analysis shows that $[\kappa] = L^2/T$. So therefore the time for information to propagate is $T \propto L^2/\kappa$. This means that, for a given resolution length Δx, numerical solutions will only be stable with a timestep $\Delta t \lesssim \Delta x^2/\kappa$. Since this scales $\propto h^2$ it quickly becomes prohibitive at high spatial resolution, unless implicit time-stepping schemes are employed.

Hyperbolic. Example: The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \tag{30}$$

Here there are second derivatives in space and time. The corresponding physics is of propagating *waves*. Since $[c] = L/T$ the time for information to propagate is $T \propto L/c$. So for a given resolution length Δx, numerical solutions will only be stable with a timestep $\Delta t \lesssim \Delta x/c$. Since this scales $\propto h$ it is the most efficient to solve in terms of timestep size.

Do penguins get cold? If a tree falls in a forest and nobody hears it, does it make a sound? What nationality is Les Murray? These are life's mysteries. But a mystery we can solve is: *Which of the above kinds of partial differential equations are the fluid equations?*

1.4 Linear solutions

1.4.1 Linear perturbation analysis

To answer the previous question, we need to reduce our set of first order partial differential equations into a single second order equation. We first assume perturbations around some constant background state ρ_0, $\boldsymbol{v}_0$, P_0, according to

$$\rho = \rho_0 + \delta\rho, \tag{31}$$

$$\boldsymbol{v} = \boldsymbol{v}_0 + \delta\boldsymbol{v} = \delta\boldsymbol{v}, \tag{32}$$

$$P = P_0 + \delta P, \tag{33}$$

where in the second line we assumed $\boldsymbol{v}_0 = 0$, implying a background state at rest. We further define the ratio between the pressure and density perturbations according to

$$c_\mathrm{s}^2 \equiv \frac{\delta P}{\delta\rho}, \tag{34}$$

which has the dimensions of a speed squared, the meaning of which will become clear. For the moment this is just a definition. Perturbing the equations, we have

$$\frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \delta\boldsymbol{v}\cdot\nabla(\rho_0 + \delta\rho) = -(\rho_0 + \delta\rho)\nabla\cdot(\delta\boldsymbol{v}), \tag{35}$$

$$(\rho_0 + \delta\rho)\left[\frac{\partial\delta\boldsymbol{v}}{\partial t} + (\delta\boldsymbol{v}\cdot\nabla)\delta\boldsymbol{v}\right] = -\nabla\delta P. \tag{36}$$

The derivatives $\partial\rho_0/\partial t$ and $\nabla\rho_0$ are zero because we assumed a constant background state. We then assume that perturbations to the background state ($\delta\rho$, $\delta\boldsymbol{v}$) are small, implying that terms involving two sets of perturbations, such as $\delta\boldsymbol{v}\cdot\nabla\delta\rho$ or $\delta\rho\nabla\cdot\delta\boldsymbol{v}$ are doubly small and hence negligible. This is obviously only true for a small perturbation to the density or velocity, but is *not* true in general. Linearising our equations this way allows us to solve them analytically, providing a useful insight into the underlying physics.

Neglecting second order terms, and using our definition (34) to relate δP to $\delta\rho$ we have

$$\frac{\partial\delta\rho}{\partial t} = -\rho_0\nabla\cdot(\delta\boldsymbol{v}), \tag{37}$$

$$\rho_0\frac{\partial\delta\boldsymbol{v}}{\partial t} = -c_{\mathrm{s},0}^2\nabla\delta\rho. \tag{38}$$

Notice that we should also assume $c_\mathrm{s} = c_{\mathrm{s},0} + \delta c_\mathrm{s}$, but that the δc_s term would be doubly small when multiplying the density perturbation. Hence why we write $c_{\mathrm{s},0}$ in (38) and also brought it out the front of the gradient term.

We desire to obtain a single equation in just one of the variables $\delta\rho$ or $\delta\boldsymbol{v}$. We can eliminate the $\delta\boldsymbol{v}$ terms by taking $\partial/\partial t$ (37), and $\nabla\cdot$ (38). This gives

$$\frac{\partial^2\delta\rho}{\partial t^2} = -\rho_0\frac{\partial}{\partial t}\nabla\cdot(\delta\boldsymbol{v}), \tag{39}$$

$$\rho_0\nabla\cdot\frac{\partial\delta\boldsymbol{v}}{\partial t} = -c_{\mathrm{s},0}^2\nabla^2\delta\rho. \tag{40}$$

Using the second equation on the right hand side of the first, we obtain

$$\frac{\partial^2\delta\rho}{\partial t^2} = c_{\mathrm{s},0}^2\nabla^2\delta\rho. \tag{41}$$

You should immediately notice that this is just the wave equation (30) expressed in terms of the density perturbation. Hence small perturbations in density travel as *waves* with speed c_s, making the meaning of our definition (34) clear. Since physically these waves are sound waves, we refer to c_s as the *sound speed*.

We can also now answer the mystery: The Euler equations are *hyperbolic*. Also yes, penguins get cold, yes, and Les Murray was Hungarian.

1.4.2 Solving the wave equation

We can solve the wave equation by assuming solutions of the form

$$\delta\rho = De^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}, \tag{42}$$

where D is the (constant) perturbation amplitude, $\boldsymbol{k}$ is the wave vector and ω is the angular frequency. Taking time derivatives with the assumed form we find

$$\frac{\partial \delta\rho}{\partial t} = -i\omega D e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = -i\omega\delta\rho, \tag{43}$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} = i^2\omega^2 D e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = -\omega^2\delta\rho. \tag{44}$$

Similarly, for the spatial derivatives we have

$$\nabla\delta\rho = ikDe^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = ik\delta\rho, \tag{45}$$

$$\nabla^2\delta\rho = i^2k^2De^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = -k^2\delta\rho. \tag{46}$$

Using (44) and (46) in (41) we have

$$-\omega^2\delta\rho = -c_{\mathrm{s},0}^2 k^2\delta\rho. \tag{47}$$

Dividing both sides by $-\delta\rho$ gives the so called *dispersion relation*

$$\omega^2 = c_{\mathrm{s}}^2 k^2. \tag{48}$$

relating angular frequency to wavenumber. Since c_{s} is positive, the solutions assuming propagation in the x direction $\boldsymbol{k} = [k_x, 0, 0]$ are given by

$$\omega = \pm c_{\mathrm{s}} k_x, \tag{49}$$

giving solutions for the density perturbation in the form of *travelling waves*

$$\delta\rho = D\exp[ik_x(x \pm c_{\mathrm{s}}t)] = D\cos[k_x(x \pm c_{\mathrm{s}}t) + \phi_0], \tag{50}$$

where ϕ_0 is some arbitrary initial phase. That is, the solution is just a sinusoidal perturbation translated, or *travelling*, to the left or right at speed c_{s}.

1.4.3 Characteristics

Another way to think about the solution (50) is that in a coordinate system with $x' = x + c_{\mathrm{s}}t$ or $x' = x - c_{\mathrm{s}}t$, or equivalently where the observer is moving with $\mathrm{d}x'/\mathrm{d}t = \pm c_{\mathrm{s}}$, the density perturbation would remain unchanged. These curves (Figure 1) are known as the *characteristics* of the wave equation and are a defining feature of hyperbolic partial differential equations.

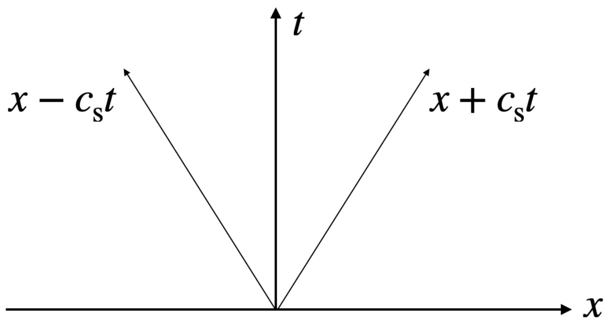

Figure 1: Characteristic curves for the wave equation, assuming a perturbation initially at $x = 0$. The density perturbation is constant along characteristics.

1.4.4 Sound speed in an adiabatic gas

If we assume an adiabatic equation of state $P = (\gamma - 1)\rho u$ with no change in entropy (i.e. $\delta u = P/\rho^2 \delta\rho$) we have

$$\begin{aligned} \delta P &= (\gamma - 1) u \delta\rho + (\gamma - 1)\rho\delta u, && (51)\\ &= \left[\frac{P}{\rho} + (\gamma - 1)\frac{P}{\rho^2}\right]\delta\rho, && (52)\\ &= \frac{\gamma P}{\rho}\delta\rho, && (53) \end{aligned}$$

Since we defined the sound speed via $c_s^2 \equiv \delta P/\delta\rho$, this implies that the sound speed in an adiabatic gas is given by

$$c_s = \sqrt{\frac{\gamma P}{\rho}}. \tag{54}$$

1.5 Non-linear solutions

1.5.1 Steepening and shock formation

The best way to think about the non-linear behaviour of the fluid equations is to consider the terms in (5)–(7) that we neglected during our linear analysis. In particular, the most important term we dropped in the linear analysis is the $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ term. We can consider this term alone while neglecting the other terms by considering the simpler equation

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = 0. \tag{55}$$

This is known as *Burgers equation*. This equation becomes trivial when expressed in terms

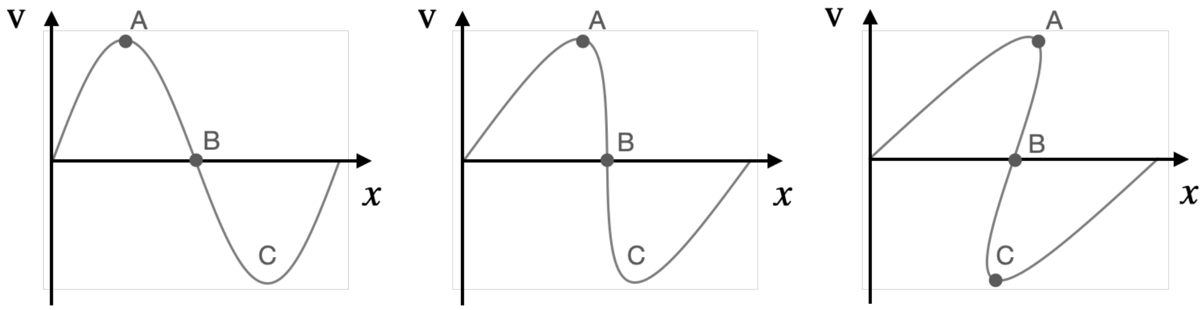

Figure 2: Non-linear behaviour of Burgers equation (56). Each Lagrangian observer A, B and C simply retains their initial velocity, which is positive for A, zero for B and negative for C. This results in steepening of the wave profile (centre panel) and ultimately a double-valued velocity field (right panel). Since the latter is unphysical in gas the end result is the formation of a discontinuity or shock wave.

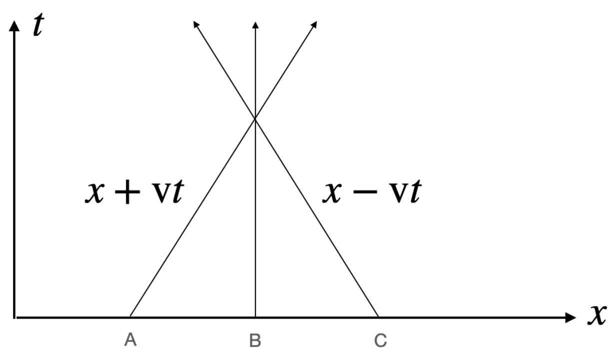

Figure 3: Characteristics for the example shown in Figure 2. The point at which characteristics cross corresponds to the formation of a shock if the fluid is collisional.

of the Lagrangian derivative, giving simply

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = 0. \tag{56}$$

That is, the velocity is constant for an observer moving with $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{v}$. This should already seem familiar, it is the same statement that we have seen for our wave solution, indicating that the velocity is constant along the characteristics of Burgers equation. We refer to quantities that are constant along characteristics as *Riemann invariants*.

We can visualise the non-linear behaviour by considering, as previously, a sinusoidal perturbation (Figure 2, left). Consider three Lagrangian observers A, B and C located at the maximum, zero and minimum of the sine wave, respectively. Since each observer simply maintains their initial velocity, one may observe that the wave starts to change shape, or *steepen* (Figure 2, centre). At some later time A and C will overtake each other resulting in a double-valued velocity field (Figure 2, right). Whether or not this occurs in practice depends on the microphysics of the fluid. In a gas or liquid at this point molecules would physically collide, resulting in dissipation and loss of energy. But there are plenty of situations where the double valued solution is the right one, namely when the fluid is *collisionless*. Examples of collisionless fluids are stars in a galaxy[4], cold dark matter, or large dust grains in protoplanetary discs. In this case one would almost always model the fluid as we have just done, namely with Lagrangian particles.

Figure 3 shows the same situation expressed in terms of characteristics. The point at which characteristics cross corresponds to formation of a shock in the gas, which is just a fancy name for a solution which is *discontinuous*. The problem with discontinuities is that it violates our assumption of having *differential equations*. That is, at the point of shock formation, the derivatives right hand sides of (5)–(7) become infinite, and our equations are no longer solvable in their present form! How can we proceed?

1.5.2 Integral vs differential form

The key is to remember that our set of differential equations is not the whole story. These equations just express higher principles, namely the conservation of mass, momentum and energy. We can apply these same principles to find solutions even when our differential equations would seem to fail us. Essentially we need to *integrate* our equations to remove the spatial derivatives. We can achieve this by writing our equations in the general 'conservative' form

$$\frac{\partial}{\partial t}(\text{thing}) + \nabla \cdot (\textbf{flux of thing}) = 0. \tag{57}$$

We already saw that the continuity equation can be written in this form, namely

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{58}$$

where ρ is the mass per unit volume and hence $\rho\boldsymbol{v}$ is the mass flux through a unit volume. Integrating this equation over a volume V gives

$$\frac{\partial}{\partial t}\int_V \rho \mathrm{d}V + \int_V \nabla \cdot (\rho \boldsymbol{v}) \mathrm{d}V = 0. \tag{59}$$

We can then use Gauss' theorem to write the second term as a surface integral, giving

$$\frac{\partial}{\partial t}\int_V \rho \mathrm{d}V = -\oint_{\partial V} \rho \boldsymbol{v} \cdot \mathrm{d}S. \tag{60}$$

[4] Galaxies are mostly empty space. So when galaxies collide the stars themselves do not actually collide, the galaxies just pass right through each other while exerting mutual gravitational forces. Nevertheless there are so many stars in a galaxy it is valid to model them as a continuous density field, i.e. as a collisionless fluid.

Physically this simply expresses the fact that mass in a volume only changes because of the mass flux in or out of the bounding surface. The key consideration for us is that there are no spatial derivatives in (60), so the equation expressed in this 'integral form' should have no problems with discontinuous solutions, unlike (58).

1.5.3 Euler equations in conservation form

Applying the same logic to the momentum and energy equations, we find that ()–() written in conservation form become

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{61}$$

$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla \cdot (P\boldsymbol{I} + \rho \boldsymbol{v}\boldsymbol{v}) = 0, \tag{62}$$

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot [(\rho e + P)\boldsymbol{v}] = 0, \tag{63}$$

where $e \equiv \frac{1}{2}v^2 + u$ is the specific total energy and $\boldsymbol{I}$ is the identity matrix. Notice that the quantity $\boldsymbol{v}\boldsymbol{v}$ in the momentum equation is actually a tensor rather than a vector[5].

Using Gauss' theorem on the flux terms, our equations in integral form are given by

$$\frac{\partial}{\partial t}\int_V \rho \,\mathrm{d}V = -\oint_{\partial V} \rho \boldsymbol{v} \cdot \mathrm{d}S, \tag{64}$$

$$\frac{\partial}{\partial t}\int_V \rho \boldsymbol{v} \,\mathrm{d}V = -\oint (P\boldsymbol{I} + \rho \boldsymbol{v}\boldsymbol{v}) \cdot \mathrm{d}S, \tag{65}$$

$$\frac{\partial}{\partial t}\int_V \rho e \,\mathrm{d}V = -\oint (\rho e + P)\boldsymbol{v} \cdot \mathrm{d}S. \tag{66}$$

1.5.4 Shock jump conditions

The simplest case is a stationary shock, for which the left hand side of the equations are zero. In this case we just have constant flux across the discontinuity and simply need to match conditions on either side of the jump. If we consider a one dimensional discontinuous jump in ρ, v_x and u, with region "1" on one side of the jump, and region

[5] We have adopted dyadic notation '$\boldsymbol{v}\boldsymbol{v}$' but often it becomes convenient to switch to tensor notation. An ugly alternative is to use outer product notation. The choice is a matter of style and convenience, but don't ask me to referee your papers if you choose the latter.

"2" on the other side, then the conditions are

$$\rho_1 v_1 = \rho_2 v_2, \tag{67}$$

$$P_1 + \rho_1 v_1^2 = P_2 + \rho_2 v_2^2, \tag{68}$$

$$\rho_1 \left(\frac{1}{2}v_1^2 + u_1 + \frac{P_1}{\rho_1}\right) v_1 = \rho_2 \left(\frac{1}{2}v_2^2 + u_2 + \frac{P_2}{\rho_2}\right) v_2. \tag{69}$$

1.5.5 Adiabatic shocks

Assuming an adiabatic equation of state $P = (\gamma - 1)\rho u$ and using (67) gives the energy jump condition (69) in the form

$$\frac{1}{2}v_1^2 + \frac{\gamma P_1}{(\gamma-1)\rho_1} = \frac{1}{2}v_2^2 + \frac{\gamma P_2}{(\gamma-1)\rho_2}. \tag{70}$$

We can then combine equations to try to solve for the density jump across the shock. From (67) we have

$$v_2^2 = \frac{\rho_1^2}{\rho_2^2} v_1^2. \tag{71}$$

Inserting this into (68) gives

$$P_1 + \rho_1 v_1^2 = P_2 + \frac{\rho_1^2}{\rho_2^2} v_1^2. \tag{72}$$

and hence

$$P_2 = P_1 + \rho_1 v_1^2 \left(1 - \frac{\rho_2}{\rho_1}\right). \tag{73}$$

Finally, inserting this into (69) gives

$$\frac{1}{2}v_1^2 + \frac{\gamma P_1}{(\gamma-1)\rho_1} = \frac{1}{2}\frac{\rho_1^2}{\rho_2^2}v_1^2 + \frac{\gamma}{(\gamma-1)\rho_2}\left[P_1 + \rho_1 v_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)\right], \tag{74}$$

$$\therefore \frac{1}{2}v_1^2 \left(1 - \frac{\rho_1^2}{\rho_2^2}\right) + \frac{c_{s,1}^2}{(\gamma-1)} = \frac{\rho_1}{\rho_2(\gamma-1)} c_{s,1}^2 + \frac{\gamma}{(\gamma-1)} \frac{\rho_1}{\rho_2} v_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right). \tag{75}$$

Multiplying by $2(\gamma - 1)$ and collecting terms gives

$$(\gamma-1)v_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right)\left(1 + \frac{\rho_1}{\rho_2}\right) + 2c_{s,1}^2 \left(1 - \frac{\rho_1}{\rho_2}\right) = \frac{2\gamma\rho_1}{\rho_2} v_1^2 \left(1 - \frac{\rho_1}{\rho_2}\right). \tag{76}$$

Cancelling the common factor, dividing by $c_{s,1}^2$ and defining the upstream Mach number $\mathcal{M}_1^2 = v_1^2/c_{s,1}^2$ we find

$$\mathcal{M}_1^2(\gamma-1)\left(1+\frac{\rho_1}{\rho_2}\right)+2 = \frac{2\gamma\rho_1}{\rho_2}\mathcal{M}_1^2,$$

$$\therefore \mathcal{M}_1^2(\gamma-1)+2 = \frac{\rho_1}{\rho_2}\mathcal{M}_1^2\left[2\gamma-(\gamma-1)\right], \tag{77}$$

$$= \frac{\rho_1}{\rho_2}\mathcal{M}_1^2(\gamma+1), \tag{78}$$

giving our final expression for the density jump in the form

$$\frac{\rho_2}{\rho_1} = \frac{\mathcal{M}_1^2(\gamma+1)}{2+(\gamma-1)\mathcal{M}_1^2}. \tag{79}$$

For a very strong shock $\mathcal{M}_1 \gg 1$ and we have

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)}{2/\mathcal{M}_1^2+(\gamma-1)} \to \frac{(\gamma+1)}{(\gamma-1)} \text{ as } \mathcal{M}_1 \to \infty. \tag{80}$$

For a monatomic gas ($\gamma = 5/3$) this implies that the maximum density jump for an infinite strength shock is

$$\frac{\rho_2}{\rho_1} = \frac{(\frac{5}{3}+1)}{(\frac{5}{3}-1)} = 4. \tag{81}$$

For a diatomic gas such as air, $\gamma = 1.4$ and hence $\rho_2/\rho_1 \to 6$ as $\mathcal{M}_1 \to \infty$.

A little more algebra shows that one can also write down the pressure and temperature jumps according to

$$\frac{P_2}{P_1} = \frac{2\gamma\mathcal{M}_1^2-(\gamma-1)}{\gamma+1}, \tag{82}$$

and

$$\frac{T_2}{T_1} = \frac{\left[2\gamma\mathcal{M}_1^2-(\gamma-1)\right]\left[2+(\gamma-1)\mathcal{M}_1^2\right]}{(\gamma+1)^2\mathcal{M}_1^2}. \tag{83}$$

Taking the limit $\mathcal{M}_1 \to \infty$ shows that both $P_2/P_1 \to \infty$ and $T_2/T_1 \to \infty$ despite the limited jump in density. Physically the limited density jump occurs because we have assumed that all of the heat generated by the shock (more on this below) is trapped. It is a completely different situation for a shock that is allowed to cool (see Section 1.5.7).

1.5.6 Shocks and irreversibility

Notice in particular that the change in kinetic energy, from (67), is given by

$$\frac{\frac{1}{2}v_2^2}{\frac{1}{2}v_1^2} = \frac{1}{16}. \tag{84}$$

This shows that kinetic energy is lost across the jump. In an adiabatic shock this must be converted to heat (u) since total energy is conserved. The curious thing about this is that it implies an *irreversible*, dissipative process. Yet we started with a set of equations defined to have no dissipation in them. Another way to think about this is that we assumed *no* irreversible processes — our differential form of the energy equation (13) implies zero change in entropy! But it is clear from (84) that the entropy must change across a shock. So how can irreversibility arise from fundamentally reversible and dissipationless equations? You may also recall we did not consider any explicit viscosity or other dissipation when formulating our equations. . .

Why a reversible set of equations becomes irreversible

How irreversibility arises in the fluid equations is subtle but easily understandable. Figure 2 shows what would happen in a situation where there was no physical dissipation at all — the solution would just become double-valued. The irreversibility arises when we mandate that the velocity field must remain single-valued. Physically this occurs in a gas because molecules *actually collide*, which is an irreversible process. Such collisions produce viscosity, but on large scales we can safely ignore this macroscopic viscosity. The issue in a shock is that there is an infinitely short length scale involved. What matters is that viscosity occurs on *some* length scale, usually far below the resolution scale in simulations[a]. The weird thing is that it does not matter *how* the viscosity occurs, there just needs to be *some* dissipation on *some* length scale. Conservation of energy across a shock jump requires it!

One may also think about where the entropy change arises from. When mandating a single-valued velocity field, we effectively have information loss at the shock front, and can no longer evolve the solution backwards in time to obtain our initial conditions. Information loss and irreversibility both imply an increase in entropy!

[a] This is also why in numerical codes the exact details of the shock capturing scheme are often not important, what matters is that there is *some form of dissipation* applied at the shock front.

1.5.7 Isothermal shocks

For an isothermal shock the jump conditions are

$$\rho_2 v_2 = \rho_1 v_1, \tag{85}$$

$$\rho_2 v_2^2 + P_2 = \rho_1 v_1^2 + P_1, \tag{86}$$

$$T_2 = T_1 \quad \text{or} \quad c_s^2 = \text{const}, \tag{87}$$

Proceeding as previously, the first equation gives $v_2^2 = \rho_1^2 v_1^2/\rho_2^2$. Using this in second expression and using $P_2 = c_s^2\rho_2$ and $P_1 = c_s^2\rho_1$ gives

$$\frac{\rho_1^2}{\rho_2} v_1^2 + c_s^2\rho_2 = \rho_1 v_1^2 + c_s^2\rho_1. \tag{88}$$

Dividing by $c_s^2\rho_2$ and defining $\mathcal{M}_1^2 \equiv v_1^2/c_s^2$ gives

$$\frac{\rho_1}{\rho_2}\mathcal{M}_1^2 \left(\frac{\rho_1}{\rho_2} - 1\right) = \left(\frac{\rho_1}{\rho_2} - 1\right), \tag{89}$$

where cancelling the factor in brackets gives

$$\frac{\rho_2}{\rho_1}\mathcal{M}_1^2 = 1, \tag{90}$$

and therefore

$$\frac{\rho_2}{\rho_1} = \mathcal{M}_1^2. \tag{91}$$

Hence in this case, as $\mathcal{M}_1 \to \infty$ then the density jump is also infinite. Physically this is because the fluid is allowed to radiate all of the heat that is generated at the shock front if we assume an isothermal equation of state.

It can be useful to consider adiabatic and isothermal equations of state to model the two extremes in any realistic astrophysical environment — adiabatic assumes all heat is trapped while isothermal assumes all heat is radiated. Just like in politics, the truth is in between.

1.5.8 Riemann invariants

A final question is to ask: What are the characteristics and Riemann invariants for the full set of fluid equations? That is, not just for Burgers equation or the linearised equations.

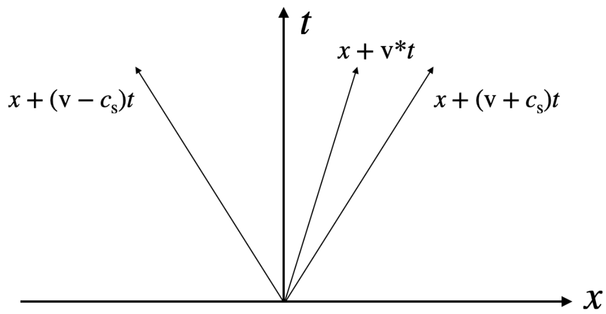

Figure 4: Characteristic curves for the fluid equations in 1D, assuming a perturbation initially at $x = 0$. A disturbance excites a left-going sound wave and a right-going sound wave. A contact discontinuity propagates at the post-shock speed.

We start with the fluid equations in the form

$$\frac{\partial \rho}{\partial t} + (\boldsymbol{v} \cdot \nabla)\rho + \rho(\nabla \cdot \boldsymbol{v}) = 0, \tag{92}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \frac{\nabla P}{\rho} = 0. \tag{93}$$

Using $P = K\rho^\gamma$ and $c_\mathrm{s}^2 = \gamma P/\rho$ we have

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} = \frac{2}{\gamma - 1}\frac{1}{c_\mathrm{s}}\frac{\partial c_\mathrm{s}}{\partial t}, \tag{94}$$

and similarly

$$\frac{\nabla \rho}{\rho} = \frac{2}{\gamma - 1}\frac{\nabla c_\mathrm{s}}{c_\mathrm{s}}. \tag{95}$$

Using these expressions in (92) and (93) we obtain

$$\frac{\partial}{\partial t}\left(\frac{2}{\gamma - 1}c_\mathrm{s}\right) + \boldsymbol{v} \cdot \nabla\left(\frac{2}{\gamma - 1}c_\mathrm{s}\right) - c_\mathrm{s}(\nabla \cdot \boldsymbol{v}) = 0, \tag{96}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + c_\mathrm{s}\nabla\left(\frac{2}{\gamma - 1}c_\mathrm{s}\right) = 0. \tag{97}$$

Adding Equations (96) and (97), assuming propagation in the x direction gives

$$\frac{\partial}{\partial t}\left(v_x + \frac{2}{\gamma - 1}c_\mathrm{s}\right) + (v_x + c_\mathrm{s})\frac{\partial}{\partial x}\left(v_x + \frac{2}{\gamma - 1}c_\mathrm{s}\right) = 0, \tag{98}$$

while subtracting (96) from (97) gives

$$\frac{\partial}{\partial t}\left(v_x - \frac{2}{\gamma - 1}c_\mathrm{s}\right) + (v_x - c_\mathrm{s})\frac{\partial}{\partial x}\left(v_x - \frac{2}{\gamma - 1}c_\mathrm{s}\right) = 0. \tag{99}$$

If we then define the quantities

$$Q \equiv v_x + \frac{2}{\gamma - 1} c_{\rm s}, \tag{100}$$

$$R \equiv v_x - \frac{2}{\gamma - 1} c_{\rm s}, \tag{101}$$

then our equations become

$$\frac{\partial Q}{\partial t} + (v_x + c_{\rm s}) \frac{\partial Q}{\partial x} = 0, \tag{102}$$

$$\frac{\partial R}{\partial t} + (v_x - c_{\rm s}) \frac{\partial R}{\partial x} = 0, \tag{103}$$

which are equivalent to

$$\frac{{\rm d}Q}{{\rm d}t} = 0; \qquad \frac{{\rm d}R}{{\rm d}t} = 0, \tag{104}$$

for observers moving with ${\rm d}x/{\rm d}t = (v_x + c_{\rm s})$ and ${\rm d}x/{\rm d}t = (v_x - c_{\rm s})$, respectively.

This shows that the quantities R and Q are constant, or *invariant* along characteristics. Hence these are the Riemann invariants for the fluid equations. Figure 4 shows the corresponding characteristics. Figure 5 shows an example of a 'shock tube' solution to the fluid equations, evolving from an initial discontinuous jump in density and pressure placed at $x = 0$. The three characteristics are evident in the structure of the solution — showing left and right-propagating sound waves and a contact discontinuity.

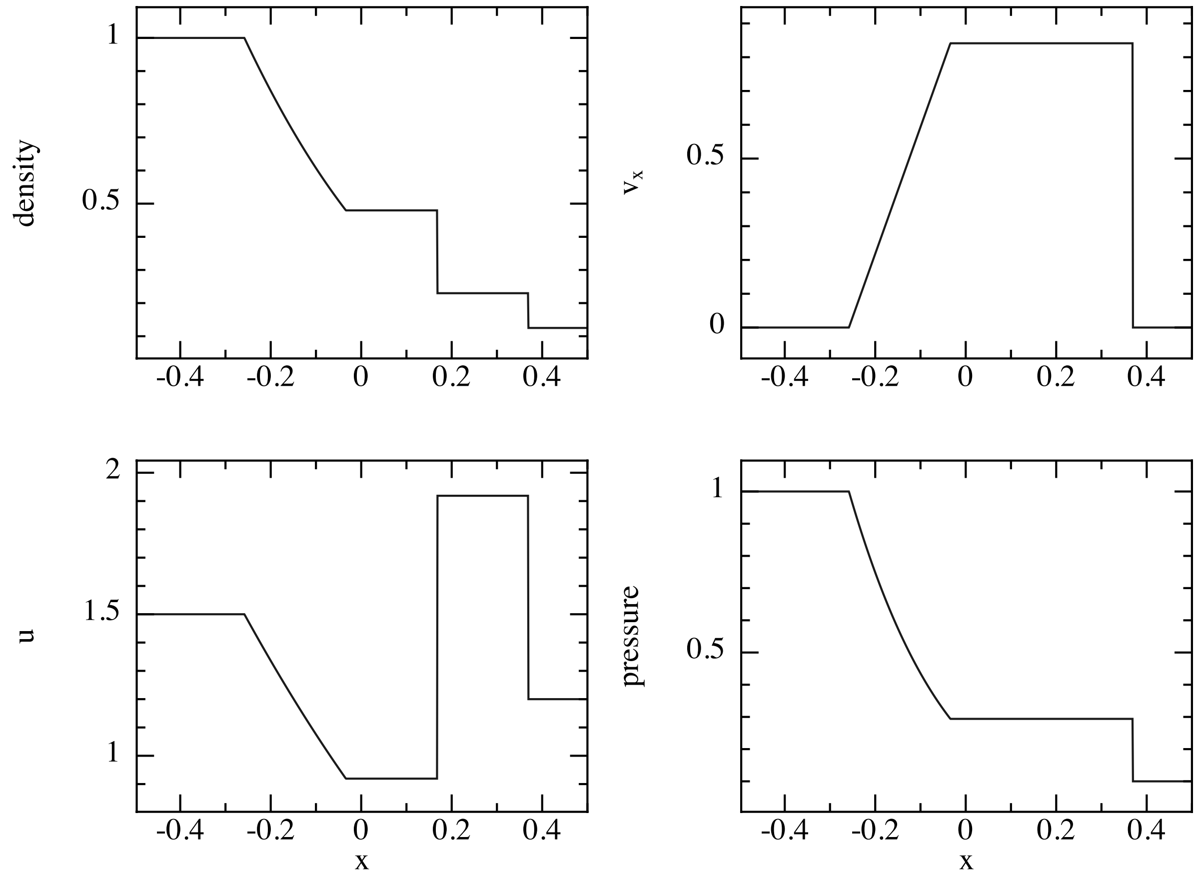

Figure 5: Exact solution for a hydrodynamic shock, where initially there was is a high pressure, high density region for $x < 0$ and a low pressure, low density region for $x \geq 0$, with a discontinuous jump at $x = 0$. A shock propagates into the undisturbed medium as the right-going sound wave, the left-going sound wave is seen as a rarefaction wave. A contact discontinuity — a jump in density and internal energy at constant pressure — propagates at the post-shock speed.

2 Dust-gas mixtures

2.1 Equation set

In astrophysics and particularly in planet formation we model the fluid as a *mixture* of gas and dust. That is we model gas and dust as two separate fluids coupled by a drag term. This is the simplest example of a 'multiphase flow'. Conservation of mass and momentum in each species generalises the equations of hydrodynamics to the following set of equations

$$\frac{\partial \rho_{\rm g}}{\partial t} + \nabla \cdot (\rho_{\rm g} \boldsymbol{v}_{\rm g}) = 0, \tag{105}$$

$$\frac{\partial \rho_{\rm d}}{\partial t} + \nabla \cdot (\rho_{\rm d} \boldsymbol{v}_{\rm d}) = 0, \tag{106}$$

$$\rho_{\rm g} \left[\frac{\partial \boldsymbol{v}_{\rm g}}{\partial t} + (\boldsymbol{v}_{\rm g} \cdot \nabla) \, \boldsymbol{v}_{\rm g} \right] = -\nabla P_{\rm g} + K \, (\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}) + \rho_{\rm g} \boldsymbol{a}_{\rm ext}, \tag{107}$$

$$\rho_{\rm d} \left[\frac{\partial \boldsymbol{v}_{\rm d}}{\partial t} + (\boldsymbol{v}_{\rm d} \cdot \nabla) \, \boldsymbol{v}_{\rm d} \right] = -K \, (\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}) + \rho_{\rm d} \boldsymbol{a}_{\rm ext}, \tag{108}$$

where K is the drag coefficient, the subscripts g and d denote gas and dust, respectively, and $\boldsymbol{a}_{ext}$ is any external acceleration term (e.g. gravity). There are two important things: First, we implicitly assumed that dust-dust collisions are sufficiently rare so that there is no pressure in the dust. Hence the pressure gradient causes acceleration of the gas but not the dust. Second, we assumed that gas and dust exchange momentum via a 'drag' term $K(\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g})$. At the moment K is just an arbitrary constant but in general would depend on fluid quantities — we will derive some possible expressions later. Key for the moment is that this term has opposite sign on the gas compared to the dust, so that the total momentum is conserved even though the gas and dust can exchange momentum with each other.

2.2 Physics of drag: The stopping time

To understand the new physics introduced by our more general set of equations, the best approach is to try to consider the effect of the drag terms in isolation. For example, consider a mixture with uniform density and velocities for each phase. In this case the spatial gradients are all zero, densities are constant in both space and time, and our equation set simplifies to

$$\frac{\partial \boldsymbol{v}_{\rm g}}{\partial t} = \frac{K}{\rho_{\rm g}}\left(\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}\right), \tag{109}$$

$$\frac{\partial \boldsymbol{v}_{\rm d}}{\partial t} = -\frac{K}{\rho_{\rm d}}\left(\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}\right). \tag{110}$$

Subtracting these gives

$$\frac{\partial(\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g})}{\partial t} = -K\left(\frac{1}{\rho_{\rm d}} + \frac{1}{\rho_{\rm g}}\right)\left(\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}\right), \tag{111}$$

Defining

$$\Delta\boldsymbol{v} \equiv \boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}, \tag{112}$$

and noticing that the prefactor on the right hand side has dimensions of inverse time, we can write this in the simple form

$$\frac{\partial \Delta\boldsymbol{v}}{\partial t} = -\frac{\Delta\boldsymbol{v}}{t_{\rm s}}, \tag{113}$$

where we see that the new physics is in the form of a new *timescale*, defined according to

$$t_{\rm s} \equiv \frac{\rho_{\rm g}\rho_{\rm d}}{K(\rho_{\rm g} + \rho_{\rm d})}. \tag{114}$$

Equation (113) is just a separable differential equation with solution (see box) given by

$$\Delta\boldsymbol{v} = \Delta\boldsymbol{v}_0 \exp\left[-\frac{(t - t_0)}{t_{\rm s}}\right]. \tag{115}$$

Solving for Δv

Assuming K independent of $\Delta \boldsymbol{v}$, this is a separable differential equation for each component, e.g. considering x direction

$$\frac{1}{\Delta v_x}\frac{\mathrm{d}\Delta v_x}{\mathrm{d}t} = -\frac{1}{t_\mathrm{s}}, \tag{116}$$

giving

$$\begin{aligned}\int_{\Delta v_x^0}^{\Delta v_x} \mathrm{d}(\ln \Delta v_x') &= -\frac{1}{t_\mathrm{s}}\int_{t_0}^{t} \mathrm{d}t' \\ \therefore \ln \Delta v_x - \ln \Delta v_x^0 &= -\frac{t-t_0}{t_\mathrm{s}} \\ \therefore \ln(\Delta v_x/\Delta v_x^0) &= -\frac{t-t_0}{t_\mathrm{s}} \\ \therefore \Delta v_x &= \Delta v_x^0 \exp\left[-\frac{(t-t_0)}{t_\mathrm{s}}\right].\end{aligned}$$

Since the solution is the same for each component, we have the vector solution in the form (115)

From the above we see that t_s is the characteristic timescale on which the differential motion between the gas and dust is reduced to zero. The two species drag each other to the barycentric (centre-of-mass) velocity given by

$$\boldsymbol{v} = \frac{\rho_\mathrm{g}\boldsymbol{v}_\mathrm{g} + \rho_\mathrm{d}\boldsymbol{v}_\mathrm{d}}{\rho_\mathrm{g} + \rho_\mathrm{d}}. \tag{117}$$

One may already notice that adding (109) and (110) simply tells us that $\partial \boldsymbol{v}/\partial t = 0$.

The solution (115) can also be used to write down the velocity of each phase. Some straightforward algebra using (117) and (112) gives

$$\boldsymbol{v}_\mathrm{g}(t) = \boldsymbol{v} + \frac{\rho_\mathrm{d}}{\rho_\mathrm{g} + \rho_\mathrm{d}}\Delta \boldsymbol{v}(t), \tag{118}$$

$$\boldsymbol{v}_\mathrm{d}(t) = \boldsymbol{v} - \frac{\rho_\mathrm{g}}{\rho_\mathrm{g} + \rho_\mathrm{d}}\Delta \boldsymbol{v}(t). \tag{119}$$

Hence we see that the velocity of each phase exponentially decays to the barycentric velocity on the timescale t_ss, which we hence refer to as the *stopping time*.

2.2.1 Stokes number

In summary, the basic physics of our new drag terms is the introduction of a new *timescale* — the stopping time. The gas and dust will drag each other towards their mutual barycentric velocity on this timescale. What is important is to consider this timescale with respect to other timescales in the problem. For dust in protoplanetary discs, the relevant timescale is the orbital period, so we define the dimensionless *Stokes number* according to

$$S_{\rm t} \equiv t_{\rm s}\Omega, \tag{120}$$

where $\Omega = \sqrt{GM/r^3}$ is the Keplerian angular speed.

2.2.2 Epstein and Stokes prescriptions for drag

To evaluate $t_{\rm s}$ we need to write down a physical prescription for the drag coefficient K. The simplest case is to consider spherical dust grains. There are two regimes:

Epstein drag Epstein (1924) considered the case when the grain size is much smaller than the mean free path of the gas, $s \lesssim \lambda_{\rm mfp}$. In this case gas molecules simply randomly bump into dust grains and bounce off in 'specular collisions'. In this case the net force on a single grain of radius s is given by

$$\boldsymbol{F}_{\rm drag} = -2\pi s^2 \rho_{\rm g} (\Delta v)^2 \left[\frac{1}{2\sqrt{\pi}} \left\{ \left(\frac{1}{\psi} + \frac{1}{2\psi^3} \right) e^{-\psi^2} + \left(1 + \frac{1}{\psi^2} - \frac{1}{4\psi^4} \right) \sqrt{\pi} {\rm erf}(\psi) \right\} \right] \Delta \boldsymbol{v}, \tag{121}$$

where $\psi \equiv \sqrt{\gamma/2} \times |\Delta v|/c_{\rm s}$. Fortunately this terrifying expression simplifies at low Mach number (the *subsonic Epstein* regime) to simply

$$\boldsymbol{F}_{\rm drag} = -\frac{4\pi}{3} \rho_{\rm g} s^2 c_{\rm s} \Delta \boldsymbol{v}. \tag{122}$$

Notice in particular that the drag on a *single* grain is proportional to its surface area ($\boldsymbol{F}_{\rm drag} \propto s^2$). For our purposes, we need to consider the *collective* drag on the fluid, so we must multiply this force by the number density of grains, in order to find the drag force *per unit volume*. The number density of grains is given by

$$n = \frac{\rho_{\rm d}}{m_{\rm d}} = \frac{\rho_{\rm d}}{\frac{4}{3}\pi s^3 \rho_{\rm gr}}, \tag{123}$$

where $\rho_{\rm gr}$ is the *intrinsic grain density*. For example $\rho_{\rm gr} \approx 3$ g/cm^3 for typical silicate grains. Thus the drag force per unit volume is given by

$$\boldsymbol{F}_{{\rm drag},V} = \boldsymbol{F}_{\rm drag} \times n = \frac{\rho_{\rm g}\rho_{\rm d} c_{\rm s}}{\rho_{\rm gr} s} \Delta \boldsymbol{v}. \tag{124}$$

Comparing this to the term in our equations, we have

$$\boldsymbol{F}_{\mathrm{drag},V} = -K\Delta\boldsymbol{v} = \frac{\rho_{\mathrm{g}}\rho_{\mathrm{d}}c_{\mathrm{s}}}{\rho_{\mathrm{gr}}s}\Delta\boldsymbol{v}, \tag{125}$$

and hence the stopping time in the subsonic Epstein regime is given by

$$t_{\mathrm{s}} \approx \frac{\rho_{\mathrm{gr}}s}{\rho c_{\mathrm{s}}}, \tag{126}$$

where $\rho \equiv \rho_{\mathrm{g}} + \rho_{\mathrm{d}}$. The key point is that the multiplication by number density means that $t_{\mathrm{s}} \propto s$; the stopping time increases with grain size.

Stokes drag occurs when the grain size exceeds the mean free path $s \gtrsim \lambda_{\mathrm{mfp}}$. In this case the grain acts like a sphere obstructing the fluid flow. In this case the stopping time is given by

$$t_{\mathrm{s}} \approx \frac{\rho_{\mathrm{gr}}s}{\rho|\Delta\boldsymbol{v}|C_{\mathrm{D}}}, \tag{127}$$

where C_{D} is a coefficient that scales according to whether or not the flow around the sphere is turbulent according to (Fassio and Probstein, 1970; Whipple, 1972)

$$C_{\mathrm{D}} = \begin{cases} 24R_{\mathrm{e}}^{-1}; & R_{\mathrm{e}} < 1, \\ 24R_{\mathrm{e}}^{-0.6}; & 1 \leq R_{\mathrm{e}} \leq 800, \\ 0.44; & R_{\mathrm{e}} > 800, \end{cases} \tag{128}$$

where the Reynolds number is defined according to

$$R_{\mathrm{e}} \equiv \frac{2s|\Delta\boldsymbol{v}|}{\nu}. \tag{129}$$

Again though, although the physics of the drag differs, the key point is that $t_{\mathrm{s}} \propto s$ also in the Stokes regime.

We find therefore that small stopping times correspond to small grains, and large stopping times correspond to large grains. This reflects our intuitive experience — smoke particles hang around in the air while rocks fall to the ground. What we define as small or large depends on the other timescales in the problem. In other words, we refer to *small grains* as those with $S_{\mathrm{t}} \ll 1$ and *large grains* as those with $S_{\mathrm{t}} \gg 1$.

2.3 Waves in a dust-gas mixture

2.3.1 Dispersion relation

Although our equation set is in general non-linear and complicated, simplified solutions provide a great deal of insight into the physics of the equations. As previously, we can obtain linear solutions by starting with perturbations of the form

$$\rho_{\rm g} = \rho_{\rm g}^0 + \delta\rho_{\rm g}, \tag{130}$$
$$\rho_{\rm d} = \rho_{\rm d}^0 + \delta\rho_{\rm d}, \tag{131}$$
$$\boldsymbol{v}_{\rm g} = \delta\boldsymbol{v}_{\rm g}, \tag{132}$$
$$\boldsymbol{v}_{\rm d} = \delta\boldsymbol{v}_{\rm d}, \tag{133}$$
$$\delta P_{\rm g} = c_{\rm s}^2\delta\rho_{\rm g}. \tag{134}$$

Substituting these into our equations and keeping only first order terms, we find

$$\frac{\partial\delta\rho_{\rm g}}{\partial t} = -\rho_{\rm g}^0(\nabla\cdot\boldsymbol{v}_{\rm g}), \tag{135}$$
$$\frac{\partial\delta\rho_{\rm d}}{\partial t} = -\rho_{\rm d}^0(\nabla\cdot\boldsymbol{v}_{\rm d}), \tag{136}$$
$$\rho_{\rm g}^0\frac{\partial\boldsymbol{v}_{\rm g}}{\partial t} = -c_{\rm s}^2\nabla\delta\rho_{\rm g} + K\left(\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}\right), \tag{137}$$
$$\rho_{\rm d}^0\frac{\partial\boldsymbol{v}_{\rm d}}{\partial t} = -K\left(\boldsymbol{v}_{\rm d} - \boldsymbol{v}_{\rm g}\right), \tag{138}$$

where for convenience we have dropped the δ when referring to velocity perturbations, simply retaining the assumption that the velocity amplitudes are small. Taking the time derivative of (135) and (136) and adding these two equations, we find

$$\frac{\partial^2\delta\rho_{\rm g}}{\partial t^2} + \frac{\partial^2\delta\rho_{\rm d}}{\partial t^2} = -\rho_{\rm g}^0\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{v}_{\rm g}) - \rho_{\rm d}^0\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{v}_{\rm d}), \tag{139}$$

while taking the divergence of (137) and (138) and adding them gives

$$\rho_{\rm g}^0\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{v}_{\rm g}) + \rho_{\rm d}^0\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{v}_{\rm d}) = -c_{\rm s}^2\nabla^2\delta\rho_{\rm g}, \tag{140}$$

which on substitution into (139) gives

$$\frac{\partial^2\delta\rho_{\rm g}}{\partial t^2} + \frac{\partial^2\delta\rho_{\rm d}}{\partial t^2} = c_{\rm s}^2\nabla^2\delta\rho_{\rm g}. \tag{141}$$

The remaining information can be extracted by subtracting (137) from (138) and taking the divergence to give

$$\rho_{\rm d}^0\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{v}_{\rm d}) - \rho_{\rm d}^0\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{v}_{\rm g}) = c_{\rm s}^2\nabla^2\delta\rho_{\rm g} - 2K(\nabla\cdot\boldsymbol{v}_{\rm d}) + 2K(\nabla\cdot\boldsymbol{v}_{\rm g}). \tag{142}$$

Substituting using (135) and (136) and their time derivatives, we find

$$-\frac{\partial^2 \delta\rho_{\rm d}}{\partial t^2} + \frac{\partial^2 \delta\rho_{\rm g}}{\partial t^2} = c_{\rm s}^2 \nabla^2 \delta\rho_{\rm g} + 2\frac{K}{\rho_{\rm d}^0}\frac{\partial \delta\rho_{\rm d}}{\partial t} - 2\frac{K}{\rho_{\rm g}^0}\frac{\partial \delta\rho_{\rm g}}{\partial t}, \tag{143}$$

giving

$$\frac{\partial^2 \delta\rho_{\rm d}}{\partial t^2} = \frac{\partial^2 \delta\rho_{\rm g}}{\partial t^2} - c_{\rm s}^2 \nabla^2 \delta\rho_{\rm g} - 2\frac{K}{\rho_{\rm d}^0}\frac{\partial \delta\rho_{\rm d}}{\partial t} + 2\frac{K}{\rho_{\rm g}^0}\frac{\partial \delta\rho_{\rm g}}{\partial t}. \tag{144}$$

Substituting this expression in (141) and dividing by two, we find

$$\frac{\partial^2 \delta\rho_{\rm g}}{\partial t^2} = c_{\rm s}^2 \nabla^2 \delta\rho_{\rm g} + \frac{K}{\rho_{\rm d}^0}\frac{\partial \delta\rho_{\rm d}}{\partial t} - \frac{K}{\rho_{\rm g}^0}\frac{\partial \delta\rho_{\rm g}}{\partial t}. \tag{145}$$

The key step to eliminating the remaining term involving $\delta\rho_{\rm d}$ is to take an additional time derivative, giving

$$\frac{\partial^3 \delta\rho_{\rm g}}{\partial t^3} = \frac{\partial}{\partial t}\left(c_{\rm s}^2 \nabla^2 \delta\rho_{\rm g}\right) + \frac{K}{\rho_{\rm d}^0}\frac{\partial^2 \delta\rho_{\rm d}}{\partial t^2} - \frac{K}{\rho_{\rm g}^0}\frac{\partial^2 \delta\rho_{\rm g}}{\partial t^2}, \tag{146}$$

upon which we can substitute using (141) to obtain

$$\frac{\partial^3 \delta\rho_{\rm g}}{\partial t^3} + K\left(\frac{1}{\rho_{\rm d}^0} + \frac{1}{\rho_{\rm g}^0}\right)\frac{\partial^2 \delta\rho_{\rm g}}{\partial t^2} - \frac{\partial}{\partial t}\left(c_{\rm s}^2 \nabla^2 \delta\rho_{\rm g}\right) - \frac{K}{\rho_{\rm d}^0} c_{\rm s}^2 \nabla^2 \delta\rho_{\rm g} = 0. \tag{147}$$

Finally, we can assume a perturbation of the form $\delta\rho_{\rm g} = De^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)}$ giving

$$\nabla^2 \delta\rho_{\rm g} = -k^2 \delta\rho_{\rm g}, \tag{148}$$

$$\frac{\partial}{\partial t}(\nabla^2 \delta\rho_{\rm g}) = i\omega k^2 \delta\rho_{\rm g}, \tag{149}$$

$$\frac{\partial^2 \delta\rho_{\rm g}}{\partial t^2} = -\omega^2 \delta\rho_{\rm g}, \tag{150}$$

$$\frac{\partial^3 \delta\rho_{\rm g}}{\partial t^3} = i\omega^3 \delta\rho_{\rm g}. \tag{151}$$

Inserting these expressions and dividing by $\delta\rho_{\rm g}$, and using our definition of $t_{\rm s}$ (noting also that $K/\rho_{\rm d}^0 \equiv \rho_{\rm g}^0/(\rho t_{\rm s})$) we find

$$i\omega^3 - \frac{\omega^2}{t_{\rm s}} - ic_{\rm s}^2 \omega k^2 + \frac{\rho_{\rm g}^0}{\rho t_{\rm s}} c_{\rm s}^2 k^2 = 0. \tag{152}$$

Multiplying by $-i/\omega$ we can write the dispersion relation in the comprehendible form

$$(\omega^2 - k^2 c_{\rm s}^2) + \frac{i}{\omega t_{\rm s}}(\omega^2 - \tilde{c}_{\rm s}^2 k^2) = 0, \tag{153}$$

where we have defined the *modified* sound speed as $\tilde{c}_{\rm s}^2 = (\rho_{\rm g}^0/\rho)c_{\rm s}^2$.

2.3.2 Interpretation

The above dispersion relation is relatively straightforward to interpret. In the limit where $t_s \to \infty$, corresponding to completely decoupled dust-and-gas ($K = 0$) the second term vanishes and we simply obtain

$$\omega^2 = c_s^2 k^2, \tag{154}$$

which is identical to the regular hydrodynamical solution, namely propagation of undamped sound waves in the gas. In the opposite limit of $t_s \to 0$, corresponding to completely coupled dust-and-gas ($K = \infty$) the first term vanishes (you can see this by multiplying the whole expression by t_s and we find

$$\omega^2 = \tilde{c}_s^2 k^2. \tag{155}$$

This corresponds to undamped sound waves propagating in a perfectly coupled *mixture*, where the only effect of dust is to slow the sound speed (multiplying it by the gas fraction). Essentially the fluid is 'weighed down' by the dust component which contributes to the inertia but not to the pressure. A key point however is that this limit of $t_s \to 0$ is perfectly well behaved and also corresponds to no damping. So both the $t_s \to 0$ and $t_s \to \infty$ limits are sensible and need to be handled correctly by numerical codes.

In between these limits we will obtain solutions for ω with imaginary components. Since $\delta\rho = De^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ having an imaginary component means that the *amplitude*s of the wave will change. This could either be instability (exponential growth of the amplitude) or damping (exponential decay of the amplitude). In the absence of other forces it turns out that the solutions to (153) always *damp* the amplitude. The strongest damping occurs when the term in the denominator is highest, namely when

$$\omega t_s \approx 1. \tag{156}$$

That is, when t_s is comparable to the wave period. This is when maximum damping of waves occurs.

In summary we learn from the linear solution that the new timescale gives us two well behaved limits with no dissipation of energy. The most interesting regime is when $S_t \approx 1$ where maximum energy dissipation takes place. In protoplanetary discs this is also when maximum settling and radial drift of grains occurs.

2.4 Dust and gas with one fluid

We have seen from the dispersion relation that both the $t_\mathrm{s} \to 0$ and $t_\mathrm{s} \to \infty$ limits are sensible and realised in nature. The $t_\mathrm{s} \to \infty$ limit is straightforward in our equation set (105)–(108): Simply uncouple the gas and dust by setting $K \to 0$ and you're done. However, the $t_\mathrm{s} \to 0$ limit is problematic, because it corresponds to $K \to \infty$ which blows up the right hand side of the equations. Another consideration is that in general with explicit time integration one will require a stability condition of the form

$$\Delta t < t_\mathrm{s}, \tag{157}$$

which means it is impossible to represent the $t_\mathrm{s} \to 0$ limit accurately with an explicit discretisation of (105)–(108).

2.4.1 Equation set

To show the limit of perfect coupling more clearly, let us change perspective. Instead of trying to describe our mixture as separate gas and dust fluids coupled by a drag term, one may consider a simple change of variables: From ρ_g, ρ_d, $\boldsymbol{v}_\mathrm{g}$ and $\boldsymbol{v}_\mathrm{d}$ to ρ, ϵ, $\boldsymbol{v}$ and $\Delta\boldsymbol{v}$ according to

$$\rho = \rho_\mathrm{g} + \rho_\mathrm{d}, \tag{158}$$

$$\epsilon = \rho_\mathrm{d}/\rho, \tag{159}$$

$$\boldsymbol{v} = \frac{\rho_\mathrm{g}\boldsymbol{v}_\mathrm{g} + \rho_\mathrm{d}\boldsymbol{v}_\mathrm{d}}{\rho_\mathrm{g} + \rho_\mathrm{d}}; \qquad \boldsymbol{v}_\mathrm{g} = \boldsymbol{v} - \frac{\rho_\mathrm{d}}{\rho}\Delta\boldsymbol{v}, \tag{160}$$

$$\Delta\boldsymbol{v} = \boldsymbol{v}_\mathrm{d} - \boldsymbol{v}_\mathrm{g}; \qquad \boldsymbol{v}_\mathrm{d} = \boldsymbol{v} + \frac{\rho_\mathrm{g}}{\rho}\Delta\boldsymbol{v}. \tag{161}$$

where ϵ is the dust fraction. Expressed in terms of these variables, equations (105)–(106) become

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\left(\nabla\cdot\boldsymbol{v}\right), \tag{162}$$

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -\frac{1}{\rho}\nabla\cdot\left(\frac{\rho_\mathrm{g}\rho_\mathrm{d}}{\rho}\Delta\boldsymbol{v}\right), \tag{163}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\frac{\nabla P_\mathrm{g}}{\rho} - \frac{1}{\rho}\nabla\cdot\left(\frac{\rho_\mathrm{g}\rho_\mathrm{d}}{\rho}\Delta\boldsymbol{v}\Delta\boldsymbol{v}\right), \tag{164}$$

$$\frac{\mathrm{d}\Delta\boldsymbol{v}}{\mathrm{d}t} = -\frac{\Delta\boldsymbol{v}}{t_\mathrm{s}} + \frac{\nabla P_\mathrm{g}}{\rho_\mathrm{g}} - \left(\Delta\boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} + \frac{1}{2}\nabla\cdot\left(\frac{\rho_\mathrm{d}-\rho_\mathrm{g}}{\rho}\Delta\boldsymbol{v}\Delta\boldsymbol{v}\right). \tag{165}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \left(\boldsymbol{v}\cdot\nabla\right). \tag{166}$$

Although we have retained the same information — we have not yet made any approximations — already the $t_s \to 0$ limit looks easier to handle. We can also see the physics of the coupling more clearly, since we have simply generalised our equation for the differential velocity evolution from (113). Notice also how (162) and (164) are just the regular fluid equations with an anisotropic pressure term and with a slight reinterpretation of quantities, since now ρ refers to the *total* mass density rather than just the gas density. Similarly, $\boldsymbol{v}$ is the barycentric velocity of the mixture rather than the gas velocity.

2.4.2 Terminal velocity approximation

We have not yet made any approximations, but we can make the t_s limit clearer by doing so. In the limit of small t_s and correspondingly small $\Delta\boldsymbol{v}$ we can neglect all but the first two terms on the right hand side of (165), giving the so-called *terminal velocity approximation* in which

$$\Delta\boldsymbol{v} \approx t_s \frac{\nabla P_g}{\rho_g}. \tag{167}$$

In this limit our equations simplify further to give

$$\frac{d\rho}{dt} = -\rho\left(\nabla\cdot\boldsymbol{v}\right), \tag{168}$$

$$\frac{d\boldsymbol{v}}{dt} = -\frac{\nabla P}{\rho}, \tag{169}$$

$$\frac{d\epsilon}{dt} = -\frac{1}{\rho}\nabla\cdot\left(\epsilon t_s \nabla P\right), \tag{170}$$

which are just the regular fluid equations with an additional equation evolving the dust fraction. The limit $t_s \to 0$ is now trivial and corresponds to simply $\epsilon =$constant, or a constant dust-to-gas ratio[6]. The downside is that the $t_s \to \infty$ limit has become correspondingly hard! However, solving (168)–(170) in numerical codes is relatively straightforward, and these equations are well behaved in the small grains (small Stokes number) limit.

[6] One may question how the equations differ *at all* from the regular fluid equations when $t_s \to 0$. The reason is in the equation of state, instead of $P = (\gamma-1)\rho u$, here we have $P = (\gamma-1)\rho_g u = \rho_g/\rho \times (\gamma-1)\rho u$. So the sound speed is modified by the gas fraction, exactly as in our dispersion relation (153).

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